On the dispersion managed soliton

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(Received 21 September 1999; accepted for publication 16 December 1999)

We review various methods used to study the dispersion managed soliton for nonlinear return-to-zero pulse propagation in optical fibers. A numerical averaging method, the guiding center soliton, the variational method both with a simple and with an extended ansatz, as well as the multiscale theory are discussed and numerically compared, allowing us to show their domains of applicability. Their relative merits and demerits are then exposed. © 2000 American Institute of Physics.

I. INTRODUCTION

Recently the so-called dispersion management method has become an essential technology for long-haul and ultrahigh-speed optical communication systems. The main purpose of dispersion management is to reduce several detrimental effects such as radiation from the pulse due to lumped amplifiers compensating fiber loss, modulational instability, jitters caused by the collisions between signals in different channels of wavelength-division-multiplexed (WDM) systems, the Gordon–Haus effect resulting from the interaction with noise, and to set a desired average value of dispersion. Additionally, the use of dispersion management leads to a possibility of upgrading installed systems (see, for example, the UPGRADE project). Several different methods have been developed to study the propagation of nonlinear return-to-zero pulses in optical fibers, called dispersion-managed (DM) solitons: the numerical averaging method, the variational theory, the guiding-center theory, and the multiscale theory.

All of these methods have successfully demonstrated their applicability to describe the several meritorious properties of the DM solitons listed above. The essence of the methods is to reduce the original perturbed system with dispersion management into a simpler equation or an approximated equation that can be rather easily solved and, more importantly, find the structure of the DM soliton. Because of their approximate nature, the methods overlook some details of the solution, and a complete theoretical understanding, such as integrability and stability, of the DM soliton remains open.

In this paper we study several aspects of the following methods: the numerical averaging method (Sec. V), the guiding-center theory (Sec. VI), the variational theory (with a Gaussian ansatz (Sec. VII B) and an extended Hermite–Gaussian ansatz (Sec. VII C)), some direct evaluation method (Sec. VIII B), and the multiscale theory in a point of view of guiding-center (or averaging) theory (Sec. IX).

By comparing numerically their results, we are able to specify their domains of applicability, i.e., the range of parameters inside which those methods can be used, such that the differences between their results and the exact solutions remain small enough (see later for what we mean by “small enough”). We show at the same time their relative precisions.

In addition, we discuss their relative complexities, to show how easy, or how difficult, it is to use them to approach the exact solution. Those merit/demerit comparisons should allow the reader to select the right method to approach the problems, depending on the time for a given precision.

Our model equation is the nonlinear Schrödinger equation (NLSE) with periodically varying coefficients:

\[ i q_z + \frac{D(Z)}{2} q_{TT} + S(Z)|q|^2 q = i G(Z)q. \]  

Here the normalized quantities \(q(Z,T), T, Z, D(Z), S(Z), \) and \(G(Z)\) express the complex envelope amplitude of the electric field \(E\), the retarded time \(t\), the propagation distance \(z\), the group velocity dispersion \(k''(z)\), the nonlinear coefficient \(\nu(z)\), and the gain (or loss) coefficient \(G(z)\) through \(q = E/\sqrt{q_0}, T = t/t_0, Z = z/z_m, \) where \(z_m = 1/\left(\nu_0q_0\right)\) and the periodic functions \(D(Z) = k''(z)z_m/t_0^2, S(Z) = \nu(z)/\nu_0, G(Z) = g(z)z_m\) with properly fixed time \(t_0\), power \(q_0\), and nonlinear coefficient \(\nu_0\). Through an appropriate choice of the
normalization $t_0$, $q_0$, and $v_0$, we can give an arbitrary value to three dimensionally independent parameters.

In this work, we use lowercase letters for quantities in real units, and uppercase letters for normalized quantities. However, the functions $q(Z,T)$ and $u(Z,T)$ are denoted by lowercase letters.

We make the usual transformation (a part of an averaging or guiding-center method),

$$q(Z,T)=u(Z,T)\exp\left\{\int_0^Z G(Z')dZ'\right\}, \quad (1.2)$$

to obtain

$$iu_2+\frac{D(Z)}{2}u_{TT}+\gamma(Z)|u|^2u=0, \quad (1.3)$$

where

$$\gamma(Z)=S(Z)\exp\left\{2\int_0^Z G(Z')dZ'\right\}. \quad (1.4)$$

Equation (1.3) now has the form of a lossless nonlinear Schrödinger equation, with $Z$-dependent coefficients, which can also be written as a Hamiltonian system with a $Z$-dependent Hamiltonian.

It should be noted that taking $\gamma$ as being constant means studying fibers with uniform nonlinearities and in which the losses can be neglected, either because they are small enough, or compensated on a much shorter scale than the dispersion. In this last case, we should see Eq. (1.3) as an equation averaged on this short scale in the sense of a guiding-center concept.

In order to simplify the comparisons of the methods exposed later, we generally take $\gamma$ as constant, although all of the methods explained here can also be applied in the non-constant (lossy) case, or the equivalent case of nonuniform nonlinearities. In this paper, all quantities cited will be in adimensional units [as in (1.3)], except when specifically stated otherwise.

The NLSE, Eq. (1.3), admits some conservation laws: three trivial ones being the energy $E=\int|u|^2\,dT$, the momentum (or center frequency) $M=(i/2)\int(\bar{u}u_T^2-u_Tu_T^2)\,dT$ and the "Hamiltonian" $H=(1/2)\int(|u_T|^4-|D(u_T|^2)\,dT$, this last one being only valid when both the dispersion and the nonlinear coefficients are constants (in which case there is an infinite number of other conserved quantities). Notice should be taken that the energy $E$ is not the real energy of the pulse, but the energy of the transformed pulse $u$ as in Eq. (1.2). This energy is constant, even in a lossy system. A test of validity of the different methods exposed here is to see whether they conserve those quantities or not (if they are applicable), since this shows then a fundamental similarity/difference in behavior with the original equation, Eq. (1.3).

II. DISPERSION MANAGEMENT FUNDAMENTALS

Although the methods exposed in this paper are more general, we applied them here only to a two-step dispersion map. Figure 1 shows the different parameters used: the average dispersion $D_{av}$, the dispersion difference $\Delta D$ and the map period $Z_t$, divided in two parts, namely, the positive dispersion fiber of length $Z_+$ and the negative dispersion fiber of length $Z_-$. We fixed the $Z$ coordinate such that $Z=0$ corresponds to the center of the positive (anomalous) dispersion fiber. A pulse propagating in a periodic dispersion map like Fig. 1 will successively broaden and compress due to the high dispersion incurred ("breathing pulse"). We are interested in periodic solutions of Eq. (1.3) in such a map for which $u(Z+Z_+,T)=u(Z,T)\exp(\imath k Z_0)$, where $Z_0=Z_++Z_-$ is the total map length and $k$ is some constant.

By appropriately choosing the factors $t_0$, $q_0$, and $v_0$ used to write Eq. (1.1), we could have fixed, for example, the average dispersion $D_{av}$, the map length $Z_0=Z_++Z_-$, and the pulse width. But making such a normalization would not allow us to study a continuous set of dispersion maps with averaging dispersion ranging from negative to positive values (but we could, for example, study the three discrete values $D_{av}=1$, $D_{av}=0$, and $D_{av}=-1$ to which any case can be rescaled). Depending on the methods used, and the cases studied, we will later fix some of those factors, or equivalently fix three adimensional parameters.

Note that although in the case where $|\Delta D|<2|D_{av}|$, the two dispersions $D_+$ and $D_-$ will have the same sign as the average dispersion $D_{av}$, we will usually say the positive (anomalous) dispersion and the negative (normal) dispersion anyway in order to simplify the discussion.

III. GENERAL PERSPECTIVE

Different methods have been developed to analyze periodic solutions of the two-step dispersion-managed problem. The first one is based on the Lagrangian theory (a variational method) with a Gaussian ansatz, and shows that solutions could also exist when $D_{av}<0$, provided that some conditions were satisfied (see Refs. 13–15, for example). Those conditions can be calculated analytically based on a small-energy calculation (see Sec. VII B 1 for details).

Another method was developed in Ref. 17 (see also Ref. 18) using a Hermite–Gaussian expansion for the solution of the NLSE Eq. (1.3), to calculate the pulse shape and the pulse evolution in a small energy case and small average dispersion.

A numerical averaging method has then been introduced in Ref. 7. It shows that exact periodic solutions seem to exist also for the NLSE, Eq. (1.3) in the fully nonlinear case (at finite energy), and permits to calculate the pulse shape of the true dispersion-managed soliton.
The variational method was refined by using a Hermite–Gaussian ansatz to give a pulse shape nearer to those purely numerical results.19 In Ref. 20, the multiple scale method is used to calculate different parameters of the pulse, in the quasilinear case (small energy) with a weak dispersion map. Reference, using also the multiple scale method, derived an equation in the quasilinear case for this pulse shape in the case of strong dispersion maps, which is equivalent to the average of the integro differential equation obtained in Refs. 21 and 22.

Although this is not the chronological order, we describe further those different methods in the following sections in another order, starting from the conceptually easiest to the more difficult ones.

IV. FREE LONG-TERM PROPAGATION

When a pulse is launched into a uniform dispersion fiber, it will generally radiate some excess energy, and turn into a soliton.

Similarly, when a pulse is launched into a two-step dispersion map, it can be shown numerically to emit some radiations. The remaining pulse presents a characteristic shape, depending only on its energy and on the dispersion map characteristics ($\Delta D, D_{av}, \gamma, Z_+, Z_-,...$).

Note that because our system is not integrable (except for the case of constant dispersion), it is difficult to separate explicitly the radiations from the signal. We call here radiations the nonperiodic part of the signal. Most of it has a velocity differing from the one of the signal, and therefore goes away from it as the pulse propagates. However, some part of it may be trapped into the signal.

Such a periodic pulse emitting no radiations seems to exist for any dispersion map with $D_{av}>0$, and for maps with $D_{av}<0$, provided that $\Delta D$ is larger than some critical value, depending on $D_{av}$.

Figure 2 shows the evolution of an initially Gaussian pulse launched into a dispersion map. The abscissa represents the propagation length, in the normalized units of Eq. (1.3). The ordinate represents the RMS pulse width at the midpoint of the positive dispersion fiber of every dispersion map ("Poincaré map").

Here we define the RMS pulse width $T_{RMS}$ and the RMS chirp $C_{RMS}$ by

$$T_{RMS}(Z) = \sqrt{\frac{\int T^2 |u(Z,T)|^2 dT}{\int |u(Z,T)|^2 dT}},$$

$$C_{RMS}(Z) = \frac{i}{4} \int \frac{TT^2 |u(Z,T)|^2 dT}{\int |u(Z,T)|^2 dT}. \quad (4.1)$$

Note that in this paper, when we do not write the limits of integration; we implicitly take them as $-\infty$ to $+\infty$.

In the first 10 000 dispersion periods, the pulse experiences large changes due to the emission of radiations, and then reaches a quasistationary state. Note that even after 50 000 periods, the pulse is not yet strictly periodic: it does still present some remaining oscillations, with a period of a few dispersion maps (this is different from the breathing inside the dispersion map, where the $T_{RMS}$ varies in this example between 1.4 and 3.7). The pulse with remaining oscillations seems to be stable around $T_{RMS} \approx 1.38$ in the Poincaré map, indicating that a fixed point corresponding to the (periodic) DM soliton is an elliptic (i.e., stable) one.

In other words, in a pulse width versus chirp graph, the pulse is orbiting around some central point, without falling on it. This indicates that the fixed point at the center of this orbit is stable: a small perturbation from this ideal point gives a small, limited orbit around that fixed point. However, a full description of the stability of the DM soliton is not known.

V. NUMERICAL AVERAGING METHOD

The free propagation method of the previous section shows that a core solution seems to exist. However, the method requires a very long simulation length to be used, and its result still presents some remaining oscillations.

Using a numerical averaging algorithm, one can exploit the fact that the phase of the core soliton is evolving regularly on one dispersion map, but the phase of the radiations may be considered to be quasirandom. Once again, we will apply this method only in our lossless case, but it can be easily applied in the lossy case, or even when third-order dispersion is present.

The original algorithm is illustrated in Fig. 3. It consists of an iteration process and allows us to calculate the stable pulse shape of a given energy $E_0$ in a given dispersion map, and from there the pulse RMS width or other relevant quantities. The iterations are initiated with an approximate pulse shape $u_0(0,T)$, which can be given by the variational method of Sec. VII, for example. A good precision on this first shape does not appear to be necessary.

One iteration consists first of calculating the evolution of this initial pulse when propagating in a few dispersion maps. Generally, some oscillations appear when the pulse width is looked at a fixed point of the periodic dispersion map (for example, $Z = m \cdot Z_1$, with an $m$ integer), i.e., in the Poincaré map. Let $Z_1$ and $Z_2$ be the distance at which two such ex-
The convergence of both of those algorithms is far superior to the simple propagation of Sec. IV, and eliminates the small oscillations that can be seen in Fig. 2. Propagation of the pulse finally obtained is then simulated in the absence of the averaging process to ensure its true periodicity. Looking only at the pulse shape at some periodic point of the dispersion map ("Poincaré map"), it is found to get back exactly to its original shape, modulus the numerical errors.

Note that this algorithm does not converge well in some cases. First, for negative average dispersion, there exist a minimal dispersion difference necessary to have a solution. If the parameters are chosen such that no solution exists, the process does not converge. Second, for low energies the system is nearly linear. In a linear system, any pulse shape is periodic if and only if \( D_{\text{av}} = 0 \). This loss of discrimination on pulse shape makes the convergence very slow at low energies. But at such energies, other theories can be used (see Sec. X). Finally, for high energies or very strong maps, convergence also becomes difficult. The reasons for this can be multiple. For once, a strong map implies that the maximum chirp is very important. The pulse is then broadened very much inside the map, requiring a very large time window to avoid losing information. But at the same time, the large chirp means a phase changing rapidly between two samples in time. The phase change between two samples must remain much smaller than \( \pi \). Those two effects combined require a very high number of time sample. The high rate of change asks also for a low step in \( z \), increasing again the computation times. Second, when the phase shift that the (nonlinear) pulse undergoes in one map approaches \( 2\pi \), there is a resonance between the pulse tail and the linear noise, which undergoes no phase shift.

The pulse evolution inside a map shows a breathing pulse, with two chirp-free points that are located at the center of the dispersion map segments in the lossless case. In Ref. 12, an analysis of the position of those chirp-free points in the quasilinear case with loss is given.

Figure 4 shows a typical pulse shape at the middle of the positive dispersion map, where the pulse has no chirp and is expressed by a real solution of the NLSE, Eq. (1.3). Using Eq. (1.3), we can find immediately that \( dT_{\text{RMS}}/dZ = 2DC_{\text{RMS}}T_{\text{RMS}} \), and hence a minimum of the pulse width is accompanied by a zero RMS chirp. This result is valid even if the dispersion \( D(Z) \) or the nonlinearities \( g(Z) \) depend on \( Z \) (see also Sec. VIII A).

The pulse of Fig. 4 presents some zeros, and the lobes that they delimit are of constant phase, and of opposite phase as their neighbors. Remark that the low \((10^{-22})\) level of
The dispersion managed soliton (frequency domain). The parameters are the same as Fig. 4.

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The true DM soliton for a nonlinear map (frequency domain). The parameters are the same as Fig. 4.

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“noise” is due to the numerical errors. The same figure also shows the chirped pulse, at its maximum chirp point inside the map, as well as its instantaneous frequency.

In a completely linear system, the pulse propagates periodically if and only if the average dispersion is zero. Its power spectrum does not change. Since the pulse may sustain here even with a finite average dispersion due to the nonlinearities, one may expect to observe a spectral change inside the dispersion map. This is illustrated in Fig. 5.

The two different dispersions in the two-step dispersion map can be given either by two different fibers, as discussed before, or by a fiber and a dispersion compensating device, like a fiber grating, for example. In this last case, the shortness of the grating makes it essentially a linear device. We have then a mixed system: a nonlinear part, followed by a linear part. We can still see periodic solutions at finite energy for a nonzero average dispersion. They present qualitatively equivalent characteristics, a major difference being that their power spectrum is now constant in the linear part of the system.

The lossy case can also be studied by the same method. The chirp-free point then differs from the map center. However, the averaging process may still be used at any point of the dispersion map (there is no need to apply it at the chirp-free point).

VI. GUIDING-CENTER SOLITON

One early technique used to study dispersion managed pulses is the guiding-center soliton theory. Developed first as a way to study the influence of the periodic amplifications (used to compensate the fiber loss) on the optical soliton in constant dispersion fibers, it has been extended to include also the effects of dispersion management. It is basically a perturbation method applied to the Schrödinger equation, Eq. (1.3), where the calculations can be carried systematically by the use of the Lie transform.

Applied to our setting, the guiding-center soliton theory considers the dispersion map as a perturbation relative to the constant (average) dispersion case, where the fundamental soliton is an exact solution. It can therefore be used near the soliton solution, and hence only when \( D_{av} \neq 0 \). (One can also extend the theory to a case near a linear pulse, which will be explained in Sec. IX.) We can then always rescale our equation to have \( D_{av} = 1 \). Rather than considering the dispersion difference \( \Delta D \) small (i.e., \( |\Delta D| \ll 1 \)) with a finite dispersion period \( [Z_i=O(1)] \), we consider the dispersion period as small (\( Z_i \ll 1 \)) and the dispersion difference as finite \( [\Delta D = O(1)] \), its effects averaging out on the small scale of the dispersion period.

The details of the general technique are exposed in Refs. 23, 26, 27. We summarize here only the results, and apply them to our specific case.

The soliton in a constant dispersion map (\( \Delta D = 0 \), with \( D_{av} = 1 \) and \( \gamma = 1 \)) is given by \( \eta \) sech(\( \eta T \exp[i\eta^2 Z/2] \)), where \( \eta \) can be chosen arbitrarily. Its energy is given by \( E = 2 \eta \) and its RMS width by \( T_{RMS} = \pi/(2\sqrt{3} \eta) \).

From Eq. (1.3), we make the change of variable \( g(Z,T) = \exp(\phi \cdot \nabla) v(Z,T) \), where \( \phi \) is called the generating function of the Lie transform. We can find the equation for \( v(Z,T) \) up to \( O(Z_i^2) \):

\[
iv_Z + \frac{1}{2} v_T + |v|^2 v = -\delta_2 [v^2 (v^\ast)^2 + (v^\ast)^2 v_T^2 + 2|v|^2 \\
\times (uv_T^\ast + v^\ast u_T + 3|v|^2)] u.
\]

For small amounts of accumulated dispersion and net amplification, i.e., \( |(D-1)Z_i|, |(\gamma-1)Z_i| \ll 1 \), the right-hand side of Eq. (6.1) can be ignored, and we have the usual averaged soliton. In the present case, this term gives an essential part of a deformation of the nonlinear Schrödinger equation, and its coefficient \( \delta_2 \) is expressed by

\[
\delta_2 = \langle \langle \langle v^2 \rangle \rangle - \langle \langle v \rangle \rangle^2 \rangle / 2 > 0,
\]

where \( v(Z) = \int_0^Z (\gamma(s) - D(s)) ds \) represents the accumulated imbalance between the dispersive and nonlinear effects, and \( \langle \langle f \rangle \rangle \) represents the weighted path average of \( f(Z) \) over a period \( Z_i \), defined by

\[
\langle \langle f \rangle \rangle = \frac{1}{Z_i} \int_0^{Z_i} f(Z) D(Z) dZ.
\]

From the Lie transform, we obtain a stationary solution for Eq. (1.3) that is valid up to \( O(Z_i^2) \):

\[
u(Z,T) = \exp[\alpha(Z,T) + i \beta(Z,T)] \nu(Z,T) + O(Z_i^2),
\]

where

\[
\nu(Z,T) = [1 + \frac{1}{2} \delta_2 \eta^4 (2 - \text{sech}^2 \tau - \text{sech}^4 \tau)] \eta \\
\times \text{sech} \tau \exp \left( \frac{i}{2} \eta^2 z \right),
\]

\[
\alpha(Z,T) = -\eta^4 [\mu(Z) - \langle \langle \mu \rangle \rangle] (\text{sech}^2 \tau)(4 - 5 \text{sech}^2 \tau),
\]

\[
\beta(Z,T) = \left( \eta^2 / 2 \right) \int_0^Z [D(s) - 1] ds
\]

\[
+ \eta^2 \langle \nu(Z) - \langle \nu \rangle \rangle \text{sech}^2 \tau.
\]

Here \( \mu(Z) = \int_0^Z (\nu(s) - \langle \langle \nu \rangle \rangle) D(s) ds \), \( \tau = \eta T \) and \( \eta \) is a free parameter that represents the pulse energy. The term \( \beta = O(Z_i) \) in the exponent of Eq. (6.4) gives a chirp
\[ \Delta \omega(Z, T) = -\partial \beta / \partial T. \] We see that it is constituted of two terms. The first term of the chirp \((\eta^2/2) \int_0^L D(s) - 1 |ds|\) results from the accumulation of the dispersion in excess, or default, relative to the average dispersion (remember that we rescaled the equations to have \(D_{av} = 1\) in this section). The second term (in \(\nu - \langle \langle \nu \rangle \rangle\)) represents the imbalance between the linear and nonlinear effects. The additional term \(a = O(Z^2_t)\) induces a periodic deformation in the pulse shape.

In our case, we have a simple two-step dispersion map, and the integrations can be easily carried away:

\[
\nu(Z) = \begin{cases} 
\frac{-\Delta D}{Z} \Delta D^2, & \text{if } -Z/4 \leq Z + nZ_1 \leq Z/4, \\
\frac{\Delta D}{2} (Z - Z/2), & \text{if } Z/4 \leq Z + nZ_1 \leq 3Z/4,
\end{cases}
\]

\[
\langle \langle \nu \rangle \rangle = 0, \quad \langle \langle \nu^2 \rangle \rangle = \frac{Z^2_t \Delta D^2}{192}, \quad \delta_\nu^2 = \frac{Z^2_t \Delta D^2}{384}. \quad (6.6)
\]

\[
\mu(Z) = \begin{cases} 
\frac{-1 + \Delta D}{2} \Delta D Z^2, & \text{if } -Z/4 \leq Z + nZ_1 \leq Z/4, \\
\frac{-1}{4} \left[ Z^2_t \Delta D^2 - 2(\Delta D/2)^2 \right], & \text{if } Z/4 \leq Z + nZ_1 \leq 3Z/4,
\end{cases}
\]

\[
\langle \langle \mu \rangle \rangle = \frac{\Delta D Z^2_t}{384} [\Delta D - 6]. \quad (6.9)
\]

Because we truncated Eq. (6.4), the energy of the solution \(u(Z, T)\) is no more completely \(Z\) independent [however, it is constant at \(O(Z^2_t)\)]. In the next discussion, when we will speak about the energy of a given solution, we will mean the energy at \(Z = 0\).

Figure 6 shows how the results of the guiding-center theory gets worse when we do not have \(Z_1 \Delta D \ll 1\). We will come back on the limits of usage in Sec. X.

Note that we used our two rescaling parameters when presenting the final results by comparing to the other methods (Sec. X). To this effect, we rescaled the equation to have \(Z_1' = 1\) from \(Z_1 \ll 1\). This implies \(D_{av}' = D_{av}, \Delta D' = \Delta D, E' = E \sqrt{Z_1}\) and \(T_{RMS}' = T_{RMS}/\sqrt{Z_1}\). This does not change the validity of the approximation, since we work at \(Z_1 \Delta D \ll E/T_{RMS}\), a condition which does not change under our rescaling.

**VII. VARIATIONAL METHOD**

The variational method developed in Ref. 10 has been widely used to study pulse propagation in an optical fiber. In this method, we assume an explicit pulse shape (the ansatz) having a few parameters and calculate their evolution, supposing that the pulse keeps its shape during propagation and can be described, along the whole dispersion map, by those finite parameters only.

In essence, we are limited to pulses representable by our ansatz. If the real evolution of the pulse is very different from what the ansatz allows, the evolution determined by those finite parameters will in reality have no meaning. When the ansatz is near to the real pulse evolution, the results are expected to provide a better approximation of the solution.

Therefore, the results of this theory should be checked against other theories, simulations and/or experiments, to assess their applicability. However, since lot of interesting results can be obtained in a rather easy and systematic manner with this method, it has been extensively used. Some of the limitations of its applicability can be overcome as shown later in this paper.

**A. General theory**

We apply the usual variational formalism to Eq. (1.3) to find the Lagrangian density \(\mathcal{L}\) and the Lagrangian \(\mathcal{L}\).

\[
\mathcal{L} = \frac{i}{2} (u_x u^* - u u_x^*) - \frac{D(Z)}{2} |u|^2 + \frac{\gamma(Z)}{2} |u|^4 \quad \text{and}
\]

\[
L = \int \mathcal{L} \, dt. \quad (7.1)
\]

To decouple the influence of the local large dispersion variations, we make the substitution

\[
u(Z, T) = A(Z) \sqrt{\rho(Z) u(Z, T)} e^{i(\tau/2) C(Z)} e^{i \theta(Z)}, \quad (7.2)
\]

with \(\tau = p(Z) T\) and where \(A(Z), p(Z), C(Z), \text{and } \theta(Z)\) are real functions. Substitution of Eq. (7.2) in Eq. (7.1) gives

\[
\mathcal{L} = A^2 p \left[ \frac{i}{2} (v_x v^* - v v_x^*) - \frac{D p^2}{2} |v|^2 + \frac{\gamma}{2} A^2 p |v|^4 \right. \\
- \tau^2 |v|^2 \left( \frac{C}{2} + \frac{D p^2 C^2}{2} \right) + i \tau (v_x v^* - v v_x^*) \\
\]

\[
\times (\rho + D p^3 C) - |v|^2 \theta, \quad (7.3)
\]

where \(\dot{x}\) denotes \(dx/dZ\).

**B. Self-similar pulses and Gaussian ansatz**

Assuming a stationary solution of the form \(u(Z, \tau) = f_0(\tau)\), we find...
\[ L = A^2 \left\{ - \frac{DP^2}{p} I_D + \frac{\gamma}{2} A^2 p I_N - \left( \frac{\dot{C}}{2} + \frac{Cp}{p} + \frac{C^2 p^2 D}{2} \right) I_C \theta I_E \right\}, \]  \tag{7.4}

where

\[ I_D = \int \left( \frac{df_0}{d\tau} \right)^2 d\tau, \quad I_N = \int (f_0^2) d\tau, \quad I_C = \int (\tau f_0^2) d\tau, \quad I_E = \int (f_0^2) d\tau. \]  \tag{7.5}

Taking the variations with respect to \( A, p, C, \) and \( \theta, \) we find

\[ \frac{dp}{dZ} = -p^3 D(Z) C, \quad \frac{dC}{dZ} = p^2 (C^2 + \alpha_1) D(Z) - \alpha_2 A^2 p \gamma(Z). \]  \tag{7.6}

\[ \frac{dA}{dZ} = 0, \quad \frac{d\theta}{dZ} = \frac{3}{2} \gamma(Z) A^2 p \frac{\alpha_3}{\alpha_E} - D(Z) p^2 \frac{\alpha_1}{\alpha_E}, \]  \tag{7.7}

where \( \alpha_1 = I_D / I_C, \quad \alpha_2 = I_N / (2 I_C), \) and \( \alpha_E = I_E / I_C. \) We can choose \( A = 1 \) [multiplying eventually \( f_0(\tau) \)], \( I_E = E_0 \) is then the pulse energy. The RMS width is given by \( T_{\text{RMS}} = \sqrt{C / E_0} \).

A Gaussian ansatz given by \( f_0(\tau) = \sqrt{E_0 / \pi} e^{-\tau^2} \) is often used because of its mathematical simplicity. We can even get reasonably good results using it. We have then

\[ I_D = \frac{E_0}{2}, \quad I_N = \frac{E_0^2}{\sqrt{2\pi}}, \quad I_C = \frac{E_0}{2}, \quad \alpha_1 = 1, \quad \alpha_2 = \frac{E_0}{\sqrt{2\pi}}. \]  \tag{7.8}

The full form for the ansatz is in this case

\[ u(Z,T) = \sqrt{p} f_0(p T) e^{(i/2)Cp^2T^2 + i\theta} \]

\[ = \sqrt{p} \sqrt{\frac{E_0}{\pi}} \exp[-p^2 T^2/2] e^{(i/2)Cp^2T^2 + i\theta}. \]  \tag{7.9}

We note here that a sech ansatz of energy \( E_0 \) of the form \( f_0(\tau) = \sqrt{E_0/2}\text{sech}(\tau) \) gives

\[ I_D = \frac{E_0}{3}, \quad I_N = \frac{E_0^2}{3}, \quad I_C = \frac{E_0\pi^2}{12}, \quad \alpha_1 = \frac{4}{\pi^2}, \quad \alpha_2 = \frac{2E_0}{\pi^2}. \]  \tag{7.10}

The periodic solutions of Eqs. (7.6) are represented in Fig. 7. Once again, solutions appear for \( D_{av} < 0 \) (normal average dispersion). Historically, this method was the first to show the existence of such solutions (see Refs. 13–15, for example). For any given dispersion difference \( \Delta D, \) a solution can be found if \( D_{av} > 0, \) but in order to have solutions when \( D_{av} < 0, \) a certain critical dispersion difference \( \Delta D_{cr}(D_{av}) \) has to be reached (\( \Delta D > \Delta D_{cr} \)). The smallest such \( \Delta D \) can be calculated by a small energy method.\(^{16}\) In order to establish this result, some general analytical results must first be proven.

The NLSE, Eq. (1.3), with constant dispersion \( D \) and nonlinearities \( \gamma \) exhibits an infinite number of conserved quantities, the first three ones being \( E = \int |u|^2 dT, \quad M = i \left( \int \left( uT^2_u - u^2 uT^2 \right) dT \right), \) and \( \mathcal{H} = \int \left( \gamma |u|^4 - D |uT^2|^2 \right) dT / 2. \) We can prove easily that, with our ansatz, the energy \( E = E_0 \) and the moment \( M = 0 \) are conserved. In addition, one can prove

\[ \mathcal{H} = - \frac{ED}{4} \left[ p^2 (\alpha_1 + C^2) - 2 \frac{\alpha_2 \gamma p}{D} \right] \]  \tag{7.11}

and

\[ \frac{d\mathcal{H}}{dZ} = - \frac{E}{4} \frac{D(Z)}{D} p^2 (\alpha_1 + C^2) + \frac{\alpha_2 p}{2} \frac{d\gamma(Z)}{dZ}, \]  \tag{7.12}

and \( \mathcal{H} \) is then conserved under Eqs. (7.6) by a section of constant dispersion and uniform nonlinearity as it is under the NLSE, Eq. (1.3), in the same conditions.

Noting \( H(Z) = -4 \mathcal{H}(Z) / (E D), \) we have

\[ \frac{dC}{dZ} = H(Z) D(Z) + \alpha_2 p(Z) \gamma(Z). \]  \tag{7.12}

\[ \Delta D = 400 \]

\[ \Delta D = 200 \]

\[ \Delta D = 80 \]

\[ \Delta D = 53.12 \]

\[ \Delta D = 20 \]

\[ \Delta D = 0 \]
Therefore, for a periodic solution, noting \( H(Z=0) = H_0 = \alpha_1 p_0^2 - 2 \alpha_2 p_0 \gamma_0 / D_+ \) and \( \Phi \cdot dZ = \int_{Z}^{Z+Z_+} \cdots dZ \), we must have \( \oint (dC/dZ) dZ = 0 \). An integral by part gives

\[
H_0 \oint D(Z) dZ + \alpha_2 \oint p(Z) F(Z) dZ = 0, \tag{7.13}
\]

where the first term is proportional to the average dispersion, and the second term is a weighted average of the peak power [proportional to \( \alpha_2 p_0(z) \)], where \( F(Z) \) depends only on the dispersion and nonlinearity profiles, and is given by

\[
F(Z) = \gamma(Z) + 2 \left[ \frac{d}{dZ} \frac{\gamma(Z)}{D(Z)} \right] \frac{Z}{\int_0^Z D(s) ds}. \tag{7.14}
\]

1. **Small energy limit**

For the case of small energy (denote \( E = \delta E \), i.e., \( \alpha_2 = \delta \alpha_2 \)), Eq. (7.13) is particularly easy to use, since, at first order, we can just take the linear solutions for \( p(Z) \), both \( \oint D(Z) dZ \) and \( E \) being first-order quantities. We find the relation between the (small) average dispersion \( \delta D_{av} \), the reduced variable \( \sigma = \sqrt{\alpha_1 p_0^2 (\Delta D/2)(Z_+ Z_-)(Z_+ + Z_-)} \) and the (small) coefficient \( \delta \alpha_2 \):

\[
\frac{\delta D_{av}}{\delta \alpha_2} = \frac{1}{\alpha_1 p_0} \frac{\gamma_+ Z_+ + \gamma_- Z_-}{Z_+ + Z_-} \left( \frac{2}{\sqrt{1 + \sigma^2}} - \frac{1}{\sigma} \arcsinh(\sigma) \right). \tag{7.15}
\]

Here we wrote the result for the case where \( \gamma \) is constant by section, taking the value \( \gamma_+ \) for \(-Z_+/2 < Z_+ + n Z_0 < Z_+/2\) and the value \( \gamma_- \) for \( Z_+ + n Z_0 < Z_+ + Z_+ \).

Remembering Eq. (7.8), and, in particular, \( \alpha_1 = 1 \) and \( \alpha_2 = E / \sqrt{2 \pi} \) for a Gaussian ansatz, we see that Eq. (7.15) gives, in fact, the angular coefficient at the origin of the curves of Fig. 7(a).

It implies that for \( \Delta D > \Delta D_{av} \), the critical dispersion difference, there exist periodic solutions in the negative average dispersion region. Such solutions are obtained for \( \delta D_{av} / \delta \Phi < 0 \), or \( \sigma < \sigma_c = 3.31982 \), which means \( \Delta D > \Delta D_{av} = 53.1171(T_{RMS}^2 / 2) Z_+ \) (for the Gaussian case when \( Z_+ = Z_+ = Z_+ / 2 \)).

C. **Extended ansatz and Hermite–Gaussian functions**

Although the variational method with a Gaussian ansatz gives interesting results, it is inherently restricted to the ansatz. If we want to find details of the pulse shape, we must refine this method. One way to do this is to take for the ansatz a series of functions, giving a high-order approximation of the pulse; that is, we take in Eq. (7.2),

\[
v(Z, \tau) = \sum_{n=0}^{\infty} a_n(Z) f_n(\tau), \tag{7.16}
\]

with \( a_n(Z) = A_n(Z) \exp(i \theta_n(Z)) \) (\( A_n \) and \( \theta_n \) being real functions).

To extend the previous Gaussian ansatz of Eq. (7.9), we consider the normalized Hermite–Gaussian functions, given by

\[
h_n(\tau; \tau_0) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n \left( \frac{\tau}{\tau_0} \right) \exp \left[ -\frac{\tau^2}{2 \tau_0^2} \right],
\]

where \( H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \),

which provide a set of complete system of functions (one complete system for every choice of the constant \( \tau_0 \)). We will determine \( \tau_0 \) later. We used the normalized \( h_n(\tau; \tau_0) \) such that

\[
\int h_n(\tau; \tau_0) h_m(\tau; \tau_0) d\tau = \tau_0 \delta_{nm}. \tag{7.18}
\]

In addition, they form the fundamental modes of the linear problem, i.e., an initially Hermite–Gaussian function (of order \( n \)) stays a Hermite–Gaussian of the same order when propagating inside any linear dispersion map.\(^{17}\)

Also, taking an input \( u(0,T) \) for the linear problem, the solution is formally given by

\[
u(Z,T) = \exp \left[ i \frac{1}{2} \left( \int_0^Z D(z') dz' \frac{\partial^2}{\partial T^2} \right) \right] u(0,T) - \frac{1}{n!} \left[ \frac{i}{2} \right]^n \left( \int_0^Z D(z') dz' \right)^n \frac{\partial^{2n}}{\partial T^n} u(0,T). \tag{7.19}
\]

If the input \( u(0,T) \) is a Gaussian pulse, the Hermite–Gaussian functions naturally appear when expanding the exponential operator.

We then take our ansatz in the form

\[
v(Z, \tau) = \sum_{n=0}^{\infty} A_n(Z) e^{i \theta_n(Z)} h_n(\tau; \tau_0). \tag{7.20}
\]

It can be proven that the variational equations issued from such an ansatz will conserve the energy \( E = \tau_0 \sum_{n=0}^{A_n(Z)} \). Note also that the momentum \( M \) of this ansatz is 0 due to the reality of \( h_n(\tau; \tau_0) \), and is thus trivially conserved.

The averaging method showed us that the stable pulse is symmetric in time \( \tau \), at its chirp-free points. In fact, the symmetry of the problem leads to \( A_{2n+1} = 0 \) in Eq. (7.20). In addition, it can be proven that an appropriate choice of \( \tau_0 \) allows us to take \( A_2 = 0 \) for the pulse shape at any given \( Z \), albeit \( \tau_0 \) being constant, nothing proves that it will remain 0 along propagation. Despite this, we took \( A_2 = 0 \). Here, \( 1/p \), that plays the role of \( \tau_0 \), will be given by the variational method. It can be shown numerically that effectively the function \( p(Z) \) obtained by the following variational equations is close to the ideal \( p(Z) \) necessary to have \( A_2(Z) \) = 0 along the map.

We then consider the ansatz of Eq. (7.20), and the truncation gives

\[
u(Z,T) = A(Z) \sqrt{\sigma_p(Z)} v(Z, \tau) \exp \left[ i C(Z) \tau^2 / 2 + i \theta(Z) \right]. \tag{7.21}
\]

\[
v(Z, \tau) = h_0(\tau) + \{ K(Z) + i S(Z) \} h_4(\tau). \tag{7.22}
\]

where \( h_0(\tau) = h_0(\tau; 1/p(z)) \) and \( h_4(\tau) = h_4(\tau; 1/p(z)) \) in Eq. (7.17). Here we write \( A_4 e^{i \theta_4} \) of Eq. (7.20) as \( K + i S \), because the coefficient \( A_4 \) might become 0, and that at such a point,
an artificial singularity is introduced into the evolution equation of $\theta_4$. Such a singularity does not appear with the later form.

We have for the energy $E_0$, the RMS pulse width $T_{\text{RMS}}$ and RMS chirp $C_{\text{RMS}}$:

$$E_0 = A^2 F_1, \quad T_{\text{RMS}} = \frac{1}{\sqrt{2} \rho} \sqrt{\frac{1 + F_2}{1 + F_1}}$$

and

$$C_{\text{RMS}} = \frac{1}{2} C \rho^2,$$

where $F_1 = 1 + K^2 + S^2$ and $F_2 = 1 + 9 K^2 + 9 S^2$. One should note that both $F_1$ and $F_2$ are functions of $Z$.

With this ansatz, the Lagrangian is given by

$$L = A^2 \left[ (SK - KS) - \frac{F_1}{2} \left( \frac{\tilde{C}}{2} - \frac{p}{p} C + \frac{Dp^2}{2} (1 + C^2) \right) \right]$$

$$+ \frac{F_1 \theta + \frac{\gamma p}{65536 \sqrt{2}} (32768 + 53760 K^2 + 17298(S^2 + K^2)^2 + 1920 S^2 + 8000 \sqrt{6K} + 192 \sqrt{6KF_1})}{1 + F_1},$$

and the variational equations are given by

$$\frac{dp}{dZ} = -CPd^3 + \frac{\gamma p^2 S}{256F_1 \sqrt{2} \pi} \left[ 3 \sqrt{6E} + 125 \sqrt{6A^2} + 1120A^2 K \right],$$

$$\frac{dC}{dZ} = DPd^2 + \frac{\gamma p}{16384 \sqrt{2} \pi} \left[ -8649E^2 + 8338EA^2 - 16073A^4 - 96 \sqrt{6EA^2} K - 4000 \sqrt{6A^4} K - 17920A^4 K^2 - 384 \sqrt{6EA^2} CS - 16000 \sqrt{6A^4} CS - 143360A^4 CKS \right],$$

$$\frac{dD}{dZ} = \frac{\gamma p S A}{2048 \sqrt{2} \pi} \left[ -3 \sqrt{6E} - 125 \sqrt{6A^2} - 1120A^2 K \right],$$

$$\frac{d\theta}{dZ} = -\frac{1}{2} DPd^2 + \frac{\gamma p}{65536 \sqrt{2} \pi} \left[ 169929E^2 + 276846E^2 - 364855A^4 + 864 \sqrt{6E^2} K + 10956 \sqrt{6EA^2} K - 93356 \sqrt{6A^2} K + 322560A^2 K^2 - 268800A^2 K^2 \right],$$

$$\frac{dK}{dZ} = 4DPd^2 + \frac{\gamma p S}{65536 \sqrt{2} \pi} \left[ -219276E^2 + 77840E^2 - 642920A^4 + 1728 \sqrt{6E^2} K + 141696 \sqrt{6EA^2} K - 160000 \sqrt{6A^2} K \right].$$

We can prove that

$$\frac{dH}{dZ} = \frac{\rho^2 A^2}{4} F_2(Z)(1 + C^2) \frac{dD(Z)}{dZ},$$

which shows that the conserved quantity $H$ of the NLSE is also conserved (by section of constant dispersion) under the variational equations, in addition to $E = E_0$ and the moment $\mathcal{M} = 0$. In addition, those equations also satisfy the momentum relations, Eqs. (8.1)–(8.2) (see Sec. VIII A).

It should be noted that although the NLSE has an infinite number of conserved quantities, nothing proves a priori that the quantities given by the same equations, where $u$ is substituted for our ansatz, will be conserved under the evolution equations given by the variational method. The fact that the first three are conserved is an encouraging factor to trust the precision of this method.

The results obtained with this extended ansatz are shown in Fig. 8, where they are compared to those of the simpler Gaussian ansatz and the averaging method. As can be seen, the Hermite–Gaussian ansatz gives better results (i.e., nearer to the averaging method), but gives also two solutions (curves $B$ and $C$) that have no correspondence in the other methods. Those spurious solutions are an artifact due to the ansatz used, and do not seem to have any physical meaning. Note that in curve $C$ only, the components in $h_0$ and $h_4$ of Eq. (7.22) are in phase at the chirp-free point $Z = 0$ (i.e., $K > 0$ and $S = 0$), where the three other curves have $K < 0$ (and $S = 0$) at this point.
Note that applied to the Gaussian ansatz, Eq. (4.1) one can easily prove the next identities:

$$
\frac{d}{dZ}T_{\text{RMS}} = 2D_{\text{RMS}}T_{\text{RMS}},
$$

(8.1)

$$
\frac{d}{dZ}C_{\text{RMS}} = \frac{D}{2} \frac{\int |u|^2 dT}{\int T^2|u|^2 dT} - \frac{\gamma}{4} \frac{\int |u|^4 dT}{\int T^2|u|^2 dT} = 4D_{\text{RMS}}^2.
$$

(8.2)

Note that applied to the Gaussian ansatz, Eq. (7.9), those two equations are equivalent to those given by the variational method, Eqs. (7.6), but are obtained without using the Lagrangian formalism. Equations (8.1)–(8.2) are also satisfied by Eqs. (7.25)–(7.30) although they are not sufficient by themself to recalculate them.

### VIII. DIRECT METHODS

#### A. Derivative evaluation

Using only Eq. (1.3), and the definition, Eq. (4.1), one can easily prove the next identities:

$$
\frac{d}{dZ}T_{\text{RMS}} = 2D_{\text{RMS}}T_{\text{RMS}},
$$

(8.1)

$$
\frac{d}{dZ}C_{\text{RMS}} = \frac{D}{2} \frac{\int |u|^2 dT}{\int T^2|u|^2 dT} - \frac{\gamma}{4} \frac{\int |u|^4 dT}{\int T^2|u|^2 dT} = 4D_{\text{RMS}}^2.
$$

(8.2)

Note that applied to the Gaussian ansatz, Eq. (7.9), those two equations are equivalent to those given by the variational method, Eqs. (7.6), but are obtained without using the Lagrangian formalism. Equations (8.1)–(8.2) are also satisfied by Eqs. (7.25)–(7.30) although they are not sufficient by themself to recalculate them.

#### B. Unstable branch of the NLSE

Since the Hermite–Gaussian ansatz of Sec. VII C gives two spurious solutions (curves B and C of Fig. 8), one can ask whether the unstable branch found with both the Gaussian and Hermite–Gaussian ansatz in the variational methods is just a spurious solution coming from the variational method, or if it corresponds to something in reality.

In order to study this, we launched into a dispersion map a series of unchirped Gaussian pulses of various RMS widths, for a fixed energy. The range of RMS widths values is chosen such that we cross the two solutions of the variational methods in Fig. 7(b).

We then measure the RMS chirp [as defined Eq. (4.1)] of the pulse after propagation along one dispersion map. In order to suppress the influence of the radiation emitted by the pulse, we measured the RMS chirp only in a window of three times the initial RMS pulse width. The results of such a measure are shown in Fig. 10.

We find effectively the two solutions when $-0.25 < D_{\text{av}} < 0$, as predicted by the variational theory.

Looking at the pulse width difference before and after one dispersion map propagation (rather than the chirp difference as in Fig. 10) gives also some zeros, near those obtained here with the RMS chirp. The fact that only a near match of the RMS widths given by those two methods is obtained is comprehensible since we do not use the periodic pulse shape (which is unknown), but only a Gaussian approximation. It shows, however, that some region of better stationarity exists, which was not found by the numerical average method, probably because such periodic (stationary) solutions would be unstable. The presence of this region is to be taken as a hint that some periodic solution, corresponding to the lower-energy branch in the negative average dispersion region seems to exist, and is not only an artifact due to the variational method. This is to be contrasted to the solutions B and C of Fig. 8 that represent such an artifact.

We plot in Fig. 11 those solutions in a $T_{\text{RMS}}$ vs $D_{\text{av}}$ graph, as well as the results of the Hermite–Gaussian ansatz and the averaging method for a comparison. It should be emphasized that those are not exact solutions, but only approximations. It shows that an initially chirp-free Gaussian pulse with a width selected according to Fig. 11 is again chirp-free after propagation on one dispersion map. Because some radiations have been emitted, its shape is no more...
Gaussian. However, is width has also nearly regained its initial value. This is why we consider that we have near stationarity (a local optimum of stationarity).

IX. MULTISCALE THEORY/GUIDING-CENTER LINEAR PULSE

The multiscale theory as applied here is a method based on the hypotheses $Z, \ll 1$, $\Delta D = O(1/Z)$, $D_{av} = O(1)$, and $u = O(1)$. We derive here the same results, based on the same hypotheses, in the point of view of the guiding-center theory. In the results of Sec. VI, we started from a conventional (i.e., constant dispersion) soliton; and took the weak dispersion maps into account as a perturbation. For strong dispersion maps, the case that we analyze here, the pulse being almost linear, we take a linear pulse as our fundamental solution, with the nonlinearities coming as a perturbation.

We define our Fourier transform by $	ilde{f}(\omega) = \int_{-\infty}^{\infty} f(T) \exp[-i\omega T] dT$, and start with the transform of Eq. (1.3), in the lossless case $[\gamma(Z) = 1]$

$$i \frac{\partial}{\partial Z} \tilde{u}(Z, \omega) - \omega^2 \bar{D}(Z) \tilde{u}(Z, \omega)$$

$$= \omega^2 \frac{D_{av}}{2} \tilde{u}(Z, \omega) - 4\pi^2 \int d\omega_1 d\omega_2 \tilde{u}(Z, \omega_1) \times \tilde{u}(Z, \omega_2) \tilde{u}^*(Z, \omega_1 + \omega_2 - \omega).$$

We have written here $D(Z) = D_{av}(Z) + \bar{D}(Z)$, where $\langle D(Z) \rangle := \int D(Z) dZ Z = D_{av}$ is the average dispersion and $\bar{D}(Z)$ is the varying part of the dispersion map. The right-hand side of Eq. (9.1) can be considered as a small perturbation to the left one.

The periodic linear solution in Eq. (9.1) (i.e., an unperturbed solution) is given by

$$\tilde{u}(Z, \omega)_1 = \exp \left[ -i \frac{\omega^2}{2} \Delta(Z) \right] \tilde{u}(0, \omega),$$

where we noted $d\Delta(Z)/dZ = \bar{D}(Z)$ and $\tilde{u}(0, \omega)$ is the initial pulse shape.

Searching a solution of Eq. (9.1) in the form

$$\tilde{u}(Z, \omega) = \exp \left[ -i \frac{\omega^2}{2} \Delta(Z) \right] \tilde{v}(Z, \omega),$$

we get the equation

$$i \frac{\partial}{\partial Z} \tilde{v}(Z, \omega) - \frac{\omega^2}{2} D_{av} \tilde{v}(Z, \omega) + \frac{1}{4\pi^2} \int d\Omega_1 d\Omega_2$$

$$\times \tilde{v}(Z, \omega + \Omega_1) \tilde{v}^*(Z, \omega + \Omega_2) \tilde{v}^*(Z, \omega + \Omega_1 + \Omega_2)$$

$$\times \exp[-i\Delta(Z)\Omega_1\Omega_2] = 0,$$

where we made the change of variable $\omega_1 = \Omega_1 + \omega$ and $\omega_2 = \Omega_2 + \omega$.

Since the large variations of $\tilde{u}(Z, \omega)$ is now absorbed into $\exp(-i\omega^2 \Delta(Z)/2)$ through the change of variable of Eq. (9.3), we can average Eq. (9.4) on one dispersion map to find the slow evolution of $\tilde{v}(Z, \omega)$. This is the key idea of the guiding-center (or averaging) theory. Here we remark that the $Z$ in $\tilde{v}(Z, \omega)$ is already a slow variable, and the only fast $Z$ dependency remaining is the one in $\Delta(Z)$.

The averaged equation over the first variable (or the guiding-center linear pulse equation) is then given by

$$i \frac{\partial}{\partial Z} \tilde{v}(Z, \omega) - \frac{\omega^2}{2} D_{av} \tilde{v}(Z, \omega) + \int d\Omega_1 d\Omega_2 \tilde{v}(Z, \omega + \Omega_1) \times \tilde{v}(Z, \omega + \Omega_2) \tilde{v}^*(Z, \omega + \Omega_1 + \Omega_2)$$

$$\times \exp[-i\Delta(Z)\Omega_1\Omega_2] = 0,$$

where the function $r(\lambda) := \int_0^Z \exp(i\lambda Z) dZ/(2\pi i)^2$, and is just a sinc function in the case of a two-step dispersion map.

Looking for traveling-wave solutions of the form $\tilde{v}(Z, \omega) = F(\omega) \exp(i\lambda Z)$ with $F(\omega)$ real and even, we find an equation for $F(\omega)$:

$$(\lambda^2 + D_{av} \omega^2) F(\omega) = 2 \int d\omega_1 d\omega_2 F(\omega + \Omega_1) F(\omega + \Omega_2) r(\Omega_1\Omega_2).$$

(9.6)

To solve numerically Eq. (9.6), we must first note that the total energy problem is decoupled from the pulse (spectrum) shape. Indeed, if $F(\omega)$ is scaled in $x \cdot F(\omega)$, the left member is multiplied by $\dot{x}$ and the right member by $x^3$, without influence of the precise shape of $F(\omega)$.

FIG. 12. A comparison between the results of the multiscale [MS] and the averaging methods [Ave]. Curves drawn for $\lambda = Z(D/8 = 1, D_{av} = 1$, and $\gamma = 1$. The graphs represent the ratio between the RMS pulse width of the solutions of those two methods, as a function of the pulse energy.
To this effect, we can use an iteration calculus. Starting with an initial pulse $F_0(\omega)$ (for example, a Gaussian), we calculate the double integral of the right member of Eq. (9.6), and divide by $\lambda^2 + D_{av}^2 \omega^2$ to find the first iteration $F_1(\omega)$. We note $F_1 = \mathcal{I}(F_0)$. Proceeding repeatedly like this ($F_{n+1} = \mathcal{I}(F_n)$), the pulse shape shows convergence when $D_{av} \geq 0$, but the pulse energy diverges. We have then to rescale the energy to some arbitrary constant at every iteration. Once the correct pulse shape is found at iteration $n$ [i.e., the left and right member of Eq. (9.6) are proportional functions], one can calculate the final pulse $F_f(\omega)$ with

$$F_f(\omega) = x \cdot F_\ell(\omega).$$

(9.8)

Figure 12 shows the comparison of this last method with the averaging method, and indicates clearly that when the condition $Z_t < 1$ ceases to be satisfied, the solutions differ. It also shows which precision can be achieved by this method as a function of the (normalized) parameters of the problem.

Rather than correcting the energy after the iterations as was shown in Eq. (9.8), this process can be integrated into the iterations. With the previous notation of $\mathcal{I}$, we take $F_{n+1} = \{\mathcal{E}(F_n)/\mathcal{E}(\mathcal{I}(F_n))\} \times \mathcal{I}(F_n)$, where $\mathcal{E}(F) = \int F^2 \, dt$ is the energy of $F$.

In the case of $D_{av} < 0$, we cannot calculate the iterations anymore, because a zero appears in the denominator of the iteration (for $\omega_a$ such that $\lambda^2 + D_{av} \omega_a^2 = 0$). However, if our spectral window (the range of frequency $\omega$ that we consider in $F$) is narrow enough to have always $\omega < \omega_a$, and/or $\lambda$ (and hence the energy) is high enough, the iteration process can be somewhat extended into the negative average dispersion region. Depending on the parameters, we could find either correct solutions, no convergence, or convergence to incorrect solutions. Those appear when the spectral window is not broad enough to capture the pulse characteristics, and are easily recognized: the energy around the extremum frequency of our window is negligible.

X. GENERAL COMPARISON

A general comparison of the numerical results of those different methods is shown in Fig. 13.

The 3% systematic error in the low-energy region of Fig. 13(a) for the Gaussian ansatz is due to the difference in pulse shape between the ansatz (a Gaussian function) and the exact solution (that has a nearly sech shape in this region). Using a sech shape for the ansatz in Eqs. (7.2)–(7.6) suppresses completely this difference at small energies, as shown in the figure.

Noting that all the methods exposed here give good results for low energies, and progressively diverge as the energy increase, we search the maximum energy at which the different methods can be used, as a function of the dispersion difference, for calculating the pulse RMS width with an error less than 5%. This is shown in Fig. 14.

Note that the Hermite–Gaussian ansatz used in the variational method gives (nearly) always the largest domain of applicability in the energy value for a fixed dispersion map.

Note also that the points where the Hermite–Gaussian ansatz seems to give a result worse than the simpler Gaussian ansatz, for $\Delta D = 10$, is illustrated in Fig. 13(a). For those values, the results of the Gaussian ansatz are oscillating around the correct value, and by chance slip outside the 5% error region for higher energies than the more steady results of the Hermite–Gaussian ansatz.

As expected, the domain of applicability of the guiding-center soliton theory becomes smaller as $\Delta D$ increases, since

![Graph (a)](image_url)

**FIG. 13.** A comparison between the results of the methods explained in this paper: averaging [(Ave)], guiding-center [(GC)], multiscale [(MS)], variational with a Gaussian ansatz [(GA)], variational with a sech ansatz [(SA)], and variational with a Hermite–Gaussian ansatz [(HG)]. Graph (a) shows the inverse of the necessary RMS width in order to have a periodic solution, as a function of the energy $E$ and the dispersion difference $\Delta D$ (calculated by the averaging method). Graph (b) represents the ratio between the RMS width obtained by various methods to the one obtained by the averaging technique for a value of $\Delta D = 10$. All curves are rescaled for $D_{av} = 1$, $Z_t$ = 1, and $\gamma$ = 1.
we consider the dispersion management as being a perturbation. Conversely, the multiple-scale method, which is asymptotically correct when $\Delta D$ is large, has a domain of applicability increasing as $\Delta D$ increases.

### XI. CONCLUSIONS

We have shown various methods widely used to study dispersion-managed solitons: the numerical averaging method (Sec. V), the guiding-center theory (Sec. VI), the variational methods [with a Gaussian ansatz (Sec. VII B) and a Hermite–Gaussian ansatz (Sec. VII C)], some direct simulation method (Sec. VII B) and the multiscale theory/the guiding-center linear pulse (Sec. IX).

One classification of those methods might be done by their precision: do we want an (asymptotically) exact result at small energy $E$ and small dispersion difference $\Delta D$ (guiding-center theory), at large $\Delta D$ (multiscale theory), a rough estimate of the RMS (Gaussian or sech ansatz in a variational method) for a limited energy range, or a good estimate on a broad energy range (Hermite–Gaussian ansatz in a variational method)? Of course, the numerical averaging method always gives the exact solution, but the calculation power needed is quite big.

This lead us to our second classification in terms of the computing power (and hence the time to get a result) as well as the computing complexity (and hence the time to program).

In this classification, the guiding-center soliton method is obviously the easiest method: it does give an explicit solution. Next is the variational methods, with a Gaussian or sech ansatz, for which we have to solve numerically a system of two ordinary differential equations (with one locally conserved quantity, reducing the system to one differential equation) with one parameter to optimize (e.g., fixing the map geometry $(Z_0, A, D_1, D_2, \gamma)$ and the initial pulse width $[T_{\text{RMS}}(Z=0)]$, and searching the pulse energy $E$ such that the solution of the system is periodic).

Next comes the Hermite–Gaussian ansatz in the variational method, which requires to solve numerically a system of five equations (with two locally conserved quantities $E$ and $H$) with two parameters to optimize (e.g., fixing again the dispersion map and the initial pulse width, we search the pulse energy $E$, and the initial $h_4$ component $K$ [see Eq. (7.22)])

The multiscale method gives a simple (i.e., short to write and conceptually easy) mathematical equation, but it is difficult to solve it. An iteration procedure was used here, with precautions to suppress a divergence in energy, as explained in Sec. IX. For every iteration, a triple integral has to be calculated, requiring a high computing power. This iteration process does also diverge for negative average dispersion.

Finally, the averaging method simulates a partial differential equation by the use of the split-step Fourier method, which requires an even higher computing power. We should note, in addition, that the convergence of this method can be very poor, especially at very low energies (we are near the linear case, for which any pulse shape will do), and at very high energies (where parasite tail oscillations appear). In the low-energy case, since the guiding-center theory gives an explicit answer, using this approximate solution as the initial pulse shape in the iterative procedure helps greatly.

In conclusion, we have summarized here the most used methods to study dispersion-managed solitons. By numerically comparing them, we have shown their energy domain of applicability; this should help one to decide which method to use, depending of what his goals are, as discussed earlier. The merits/demerits of those methods (precision, calculation power,...) have also been exposed.

13. Y. Kodama, in Ref. 6, pp. 131–154.
25. Y. Takushima and K. Kikuchi, in Conference on Lasers and Electro-