The dynamics of degenerate classical systems can have very interesting properties. Under certain conditions, an infinitely small perturbation can generate in phase space an infinite stochastic web possessing global symmetry. A particle, placed inside the web, can gain energy traveling along the web. The symmetry of the web is determined by the type of perturbation. In this paper we study a system whose classical counterpart possesses a stochastic web with the axial symmetry of the order $2\mu$, where $\mu$ is the ratio between the frequency of the perturbation and the eigenfrequency of the unperturbed system. We study the global properties of the quantum analog of the classical stochastic web and show that the quantum web possesses the same symmetry as its classical counterpart. The system is analyzed in the limit where the stochastic layers in the phase space are negligibly thin. The approach developed in this paper can be used to analyze the global properties of different quantum dynamical systems with complicated symmetries.

It is known that the phase space of a classical harmonic oscillator weakly interacting with a plane monochromatic wave possesses an interesting and nontrivial symmetry. (See, e.g., Ref. 1, and references therein.) In the case of exact resonance, $\mu \omega = \Omega (\mu = 1, 2, \ldots)$, between the wave (with the frequency $\Omega$) and the harmonic oscillator (with the oscillation frequency $\omega$), and under the condition $e \ll 1$ (where $e$ is a dimensionless perturbation parameter), the classical phase space consists of an infinite number of resonant cells arranged in a pattern with the axial symmetry of the order $2\mu$. An example of a corresponding phase space with $\mu = 4$ is shown in Fig. 1. At the center of each cell there is an elliptic stable point. The particles move in the phase space around this point along the closed trajectories. The cells are separated from each other by the separatrices, which are schematically shown in Fig. 1 by dashed lines. These separatrices form in the phase space an unlimited net. The net is covered by the stochastic layers forming the infinite stochastic web. When the perturbation parameter, $e$, is small the web width is exponentially thin. However, if the particle is initially placed inside a stochastic region, it can travel throughout the web and gain energy, even for an arbitrarily small perturbation parameter, $e$. The existence of the crystalline and quasicrystalline symmetries of the classical phase space and the stochastic web differ significantly this system from classical nonlinear systems with chaotic behavior. These interesting properties of the classical harmonic oscillator in a monochromatic wave motivated our studies of the corresponding properties in the quantum system.
acting with a plane monochromatic wave under the condition of resonance. It is shown that the Husimi functions of all Floquet states have the symmetry of the order $2\mu$, where $\mu$ is the resonance number. The quantum phase space is symmetric only in the resonance approximation when one can neglect the structure of the exponentially thin stochastic layers near the separatrices. The quantum-classical correspondence is demonstrated numerically for the Floquet states responsible for the motion in the region near the elliptic stable points (centers of the resonant cells). The developed approach can be used to recover the global symmetry of the classical phase space for a more general class of dynamical systems with the infinite stochastic web.

The quantum harmonic oscillator interacting with a monochromatic wave is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2x^2}{2} + \frac{\epsilon}{k}\cos(kx - \Omega t) = \hat{H}_0 + \hat{V}(x,t),$$

where $\hat{H}_0$ is the Hamiltonian of the harmonic oscillator, $\hat{V}(x,t)$ is the interaction Hamiltonian, $\epsilon/k$ and $k$ are, respectively, the amplitude and wave vector of the wave, $x$ and $\hat{p}$ are the coordinate and the momentum operators of the particle, $m$ is the mass of the particle. The Hamiltonian (1) appears, for example, when analyzing the stability of an ion in a linear ion trap in the field of two laser beams with close frequencies\(^8\) or studying the quantum cyclotron resonance\(^9\) in metals and semiconductor heterostructures. The dynamics of the quantum system described by the Hamiltonian (1) is controlled by four parameters: the resonance number, $\mu$; the detuning from the exact resonance: $\delta\omega = \mu\omega - \Omega$; the dimensionless perturbation parameter: $\epsilon = \epsilon/km\omega$; and the dimensionless Planck constant: $h = (\hbar k)/(m\omega)$. (For the system considered in Ref. 8, the Lamb–Dicke parameter, $\eta$, is related to $h: h = 2\eta^2$). Influence of these parameters on the dynamics of the quantum system was considered in detail in Refs. 8–11.

In this paper, we study the global properties of the “quantum phase space” of a system described by the Hamiltonian (1) for the case of exact resonance, $\delta\omega = 0$ (when the number of the resonance cells is infinite) and under the condition: $\epsilon \ll 1$ (when the chaotic layers, covering the separatrices, are exponentially thin). We investigate the structure of the “quantum phase space.” We show that the quantum system possesses the same global symmetry as the classical system.

The structure of the quantum system with the Hamiltonian (1) is characterized by the Floquet states (or quasienergy states) found in Refs. 9–11. In order to build the phase space for the quantum system we use the Husimi functions of the Floquet states. The Husimi function for an arbitrary wave function $\psi(x,t)$ is defined as the projection of $\hat{\Phi}(x,t)$ on the coherent wave packet, $\chi(x,X,P)$, with the maximum at the point $(X,P)$,\(^12\)

$$\Phi(X,P,t) = \frac{1}{2\pi} |\psi(x,t)|^2 \chi(x,X,P).$$

The Husimi function, $\Phi(X,P,t)$, defines the probability of finding a quantum particle characterized by the wave func-
tion $\psi(x,t)$ at the point $(X,P)$ of the “quantum phase space.” The cross sections of the Husimi function are the lines of equal probability of finding the quantum particle. In the following, these lines for Husimi functions are compared with the trajectories in classical phase space.

Namely, we analyze the structure of the Husimi functions of the quasienergy states and compare them with the structure of the classical phase space. First, we present some general formulas which will be used to investigate the system described by the Hamiltonian (1). It is convenient to decompose the coherent state, $\chi(x;X,P)$, into the complete set of harmonic oscillator eigenstates,

$$\chi(x;X,P) = \exp\left(-\frac{X^2 + P^2}{2h}\right) \sum_{m=0}^{\infty} \frac{(X + iP)^m}{(2h)^{m/2} m!} \psi_m(x),$$

(3)

where $\psi_m(x)$ is the $n$th eigenfunction of the harmonic oscillator with the Hamiltonian $\hat{H}_0$. In Eq. (3) we used the dimensionless coordinate $(X = kx)$ and the dimensionless momentum $(P = pk/(i\omega))$. We use the same basis to represent the wave function, $\psi(x,t)$,

$$\psi(x,t) = \sum_{n=0}^{\infty} C_n(t) \psi_n(x) \exp\left[-i\omega t \left(n + \frac{1}{2}\right)\right].$$

(4)

The structure of the Husimi function (2) is completely defined by the coefficients $C_n(t)$,

$$\Phi(X,P,t) = \frac{\exp\left(-X^2 + P^2/(2h)\right)}{2\pi h} \times \left| \sum_{m=0}^{\infty} C^*_{m}(t) \frac{(X + iP)^m}{(2h)^{m/2} m!} \exp(i\omega t) \right|^2.$$  

(5)

It is convenient to use the cylindrical coordinates,

$$X = r \cos \varphi, \quad P = r \sin \varphi,$$

(6)

where $r = \sqrt{X^2 + P^2}$ and $\varphi = \arctan(P/X)$. In these variables, the Husimi function (5) is

$$\Phi(r,\varphi,t) = \frac{\exp\left(-r^2/(2h)\right)}{2\pi} \times \left| \sum_{m=0}^{\infty} C^*_{m}(t) \frac{r^m \exp(i\omega t)}{(2h)^{m/2} m!} \right|^2.$$  

(7)

Since the perturbation, $V(x,t)$, in Eq. (1) is periodic in time, one can use Floquet theory and write the solution of the nonstationary Schrödinger equation as

$$\psi_q(x,t) = \exp(-iE_q t/h) U_q(x,t),$$

(8)

where $U_q(x,t) = U_q(x,t + T)$ is a time-periodic function whose period is $T = 2\pi/\Omega$. The index $q$ labels the quasienergy (QE) states. It is convenient to use the complete set of harmonic oscillator eigenfunctions to represent the function $U_q(x,t)$,

$$U_q(x,t) = \sum_{n=0}^{\infty} C^q_n(t) \psi_n(x),$$

(9)

where the expansion coefficients, $C^q_{m}(t) = C^q_{m}(t + T)$, are time-periodic functions. Using Eqs. (3), (8) and (9) we can rewrite the Husimi function of the QE state as

$$\Phi_q(r,\varphi,\omega t) = \frac{\exp\left(-r^2/(2h)\right)}{2\pi} \sum_{m=0}^{\infty} C^q_{m} r^m \exp(i\omega t)} \frac{1}{\sqrt{(2h)^{m/2} m!}},$$

(10)

where $m = \mu n$, $n = 0, 1, 2, \ldots$, $C^q_{m} = C^q_{m}(sT)$, $s = 0, 1, 2, \ldots$, and the Husimi functions of the Floquet states (10) are independent of $s$. The QE states of the monochromatically perturbed harmonic oscillator were studied in detail in a series of papers (Refs. 9–11), using degenerate resonance perturbation theory for the Floquet states. In particular, the quantum regimes corresponding to regular motion and to the case of weak chaos in the classical phase space were investigated. As was shown in Ref. 9, the Hilbert space of the quantum system breaks up to some approximation into the dynamically independent regions—quantum resonance cells, each of them with its own set of QE states. In the zeroth order (resonance approximation), the QE functions and the QE spectrum of each cell are almost independent. Near the top and bottom of the QE spectrum of an individual cell, the QE states are the states of an effective harmonic oscillator. The QE levels are equally spaced, with a separation $\hbar \omega$ between the levels. The frequency $\omega$, in the quasiclassical limit coincides with the frequency of small oscillations near the center of the resonance in the phase space.

In the resonance approximation, the time dependence of the QE functions, $C^q_{m}(t)$, has the form: $C^q_{m}(t) = C^q_{m} \exp(i\omega t)$, where the index $n$ takes the values $n = \mu m$ ($m = 0, 1, 2, \ldots$). As follows from Eq. (10), the dynamics of the Husimi function in the resonance approximation is a simple rotation of the phase space around the point $(X = 0, P = 0)$, with the frequency $\omega$. The QE spectrum in the resonance approximation is symmetric, and the QE functions, $C^q_{m} = C^q_{m}(sT)$, $s = 0, 1, 2, \ldots$ of the symmetric levels are connected by the relations

$$E_{\mu n} - E_q = C^q_{\mu m} \rightarrow - (1)^{m} C^q_{\mu m},$$

(11)

associated with the transformation: $x \rightarrow -x$ in Eqs. (4) and (9).

One can easily note from Eq. (10) one more important property of the system under consideration: In the resonant approximation, the Husimi function is symmetric with respect to rotation by the angle $\varphi \rightarrow \varphi + 2\pi/\mu$. The transformation (11) yields the same Husimi function but rotated by the angle $\varphi \rightarrow \varphi + \pi/\mu$. As a result, all the Husimi functions of the quasienergy states possess axial symmetry of the order $2\mu$. In other words, the quantum phase space in the resonance approximation has the same axial symmetry as its classical analog (see Fig. 1). The quantum system possesses the property of symmetry only in the resonance approximation when only terms with $m = \mu n$ are different from zero in Eq. (10).

In order to demonstrate this correspondence, let us consider the Husimi functions of the QE states corresponding to the QE levels near the edges of the spectrum. We choose these QE states for of the following reasons. The upper and
the lower QE levels of one cell are approximately equidistant, with the level separation \( h\omega \), where \( \omega \) in the quasiclassical limit becomes identical to the frequency of small oscillations near the center of the classical resonance cell in the phase space. Thus, the QE states near the edges of the spectrum of one quantum cell are the oscillator states with the distance between the QE levels, \( h\omega \), the QE eigenfunction corresponding to the upper oscillator level is the Gaussian wave packet,

\[
C_n^e = \Gamma \exp\left(-\frac{(n-n_c)^2}{2a_e^2}\right),
\]

where \( \Gamma \) is the normalization factor, and \( n_c \) is the position of the maximum of the QE wave packet in the Hilbert space which corresponds to the quantized radius of the elliptic stable point: \( r_c = \sqrt{2n_c\hbar} \) (see Ref. 10). The width of the wave packet, \( a_e \), in Eq. (12) was defined in Ref. 10 in the form

\[
a_e = \left(\frac{g_{\mu}(n_c)}{g_{\mu}''(n_c)}\right)^{1/4},
\]

where the function \( g_{\mu}(n) \) is expressed in terms of the matrix element: \( g_{\mu}(n) = \langle \phi_{\mu}\rangle \cos(x)\phi_{\mu+n}\rangle \). In the quasiclassical region of parameters, Eq. (13) can be expressed through the half width in action of the classical resonance cell, \( \Delta I \) (the expression for the value of \( \Delta I \) see, for example, in Ref. 13),

\[
a_e = \left(\frac{r_c^2}{\hbar^2} J_{\mu}(r_c)\right)^{1/4} \frac{\Delta I^{1/2}}{\sqrt{2\hbar}},
\]

where the prime indicates differentiation with respect to the argument. For example, for \( \mu = 1 \), we have \( a_e = r_c/\{r_c^2[(r_c)^2-1]\}^{1/4} \). The boundaries of the quantum cells are given by the zeros of the function \( g_{\mu}(n) \). As was shown in Ref. 9, the function \( g_{\mu}(n) \) is proportional to the Bessel function, \( J_{\mu} \), of order \( \mu \): \( g_{\mu}(n) \sim J_{\mu}(\sqrt{2n\hbar}) \). So, the number of levels in the individual cell is proportional to \( \hbar \). Thus, the ratio (the packet’s width in \( n \)) of the cell’s width in \( n \) is proportional to \( \sqrt{\hbar} \), and in the quasiclassical limit the relative width of the QE ground state tends to zero.

The Husimi representation allows one to construct the QE eigenstates in the quantum phase space. The simplest case is the Husimi function of a single harmonic oscillator state: \( C_n = \delta_{n,n_0} \), which due to Eq. (7) has the following form:

\[
\Phi^{(n_0)}(r,\varphi,t) = \frac{\exp\left(-r^2/2\hbar\right)}{2\pi\hbar} \frac{r^{2n_0}}{(2\hbar)^{n_0}n_0!},
\]

This expression has its maximum at \( r_0 = \sqrt{2n_0\hbar} \). The definite value of \( n_0 \) corresponds to the definite value of the action:

\[
I_0 = \hbar n_0. \]

Due to the fundamental uncertainty relation, the phase, \( \varphi \), of this state is indefinite. The Husimi function is independent of the phase, \( \varphi \), and looks like a round hump with the center in the point \( (X = 0, P = 0) \).

In agreement with Eqs. (10) and (12), the Husimi function of the upper QE state is

\[
\Phi_{\varphi}(r,\varphi,t) = \Phi_{\varphi}(r,\varphi) = \exp\left(-r^2/2\hbar\right) \frac{r^{2n_e}}{2\pi\hbar} \frac{(2\hbar)^{n_e}n_e!}{\sqrt{(2\hbar)^{n_e}n_e!}}
\]

\[
\times \exp\left(-\frac{(m-n_e)^2}{2a_e^2}\right) \frac{\exp\left(-n^2/2a_e^2\right)}{2\pi\hbar^{n_e}n_e!}.
\]

Only \( \Delta \sim a_e \) terms with \( |m-n_e| < a_e \) effectively contribute to the sum on the right-hand side of Eq. (16), and one can neglect all other terms. Then, Eq. (16) becomes

\[
\Phi_{\varphi}(r,\varphi,s) = \exp\left(-r^2/2\hbar\right) \frac{r^{2n_e}}{2\pi\hbar} \frac{(2\hbar)^{n_e}n_e!}{\sqrt{(2\hbar)^{n_e}n_e!}}
\]

\[
\times \exp\left(-\frac{(m-n_e)^2}{2a_e^2}\right) \frac{\exp\left(-n^2/2a_e^2\right)}{2\pi\hbar^{n_e}n_e!}.
\]

The double sum in Eq. (17) can be rewritten as

\[
\sum_{n, m = -\Delta m}^{\Delta m} \frac{r^{n+m}e^{i\phi(n-m)}}{(2\hbar n_e)^m} \exp\left(-\frac{n^2+m^2}{2a_e^2}\right)
\]

\[
= \sum_{j = -\Delta m}^{\Delta m} \frac{r^j}{(2\hbar n_e)^j} \exp\left(-\frac{j^2}{4a_e^2}\right)
\]

\[
\times \sum_{k = -\Delta m}^{\Delta m} e^{ik\varphi} \exp\left(-\frac{k^2}{4a_e^2}\right),
\]

where \( j = n + m, k = n - m \). Thus, by using the approximation (18) we find that the Husimi function of the extreme QE state can be factorized,

\[
\Phi_{\varphi}(r,\varphi,s) = \gamma(r)\xi(\varphi,s).
\]

In Eq. (19),

\[
\gamma(r) = \exp\left(-r^2/2\hbar\right) \frac{r^{2n_e}}{2\pi\hbar} \frac{(2\hbar)^{n_e}n_e!}{\sqrt{(2\hbar)^{n_e}n_e!}}
\]

\[
\times \sum_{j = -\Delta m}^{\Delta m} \frac{r^j}{(2\hbar n_e)^j} \exp\left(-\frac{j^2}{4a_e^2}\right),
\]

\[
\xi(\varphi,s) = \sum_{k = -\Delta m}^{\Delta m} e^{ik\varphi} \exp\left(-\frac{k^2}{4a_e^2}\right).
\]

Let us determine now the positions of maxima of \( \Phi_{\varphi}(r,\varphi) \). Suppose that each maximum of the Husimi func-
tion corresponds to the stable elliptic point at the center of a resonance cell. Maximum of $\Phi_q(r, \varphi)$ in $r$ is defined from

$$
\frac{d}{dr} \gamma(r) = \left[ \frac{d}{dr} \exp\left(\frac{-r^2}{2\hbar}\right) \right] \left[ \frac{r^{2n_q} |\Gamma|^2}{2\hbar^n n_q!} \right] 
\times \sum_{j=-2\Delta m}^{2\Delta m} \left( \frac{r}{r_c} \right)^j \exp\left( -\frac{j^2}{4a_e^2} \right) 
+ \exp\left(\frac{-r^2}{2\hbar}\right) \left[ \frac{r^{2n_q} |\Gamma|^2}{2\hbar^n n_q!} \right] 
\times \sum_{j=-2\Delta m}^{2\Delta m} \frac{j}{r_c} \left( \frac{r}{r_c} \right)^{j-1} \exp\left( -\frac{j^2}{4a_e^2} \right) = 0. \quad (22)
$$

When $r = r_c$, both sums in Eq. (22) are zero: In the first term, the derivative is equal to zero as follows from Eq. (15); in the second term, the sum is equal to zero, and the value $r_c$ can be considered as the radius of the center of the quantum resonance cell in the quantum phase space.

We now find the maxima of $\xi(\varphi, s)$. It is convenient to present this function in the form

$$
\xi(\varphi) = 1 + 2 \sum_{m=1}^{2\Delta m / \mu} \cos(\mu m \varphi) \exp\left( -\frac{(\mu m)^2}{4a_e^2} \right), \quad (23)
$$

where we took into account that in the resonance approximation the particle can populate only states with the numbers: $k = \mu m$ (see Ref. 9). All terms in the sum on the right-hand side of Eq. (23) decrease in absolute values as $m$ increases. Then, the extrema of the function $\xi(\varphi)$ is defined by the extrema of the term with $m = 1$. When $\mu = 1$ there is one maximum at $\varphi = 0$; when $\mu = 2$ there are two maxima at $\varphi = 0$ and $\varphi = \pi$. In the general case the function $\xi(\varphi)$ has $\mu$ maxima.

The extreme lower QE function is related to the extreme upper one by the transformation (11), which is convenient to rewrite in the form: $C^0_{\mu m} \rightarrow \exp(-i\pi m) C^0_{\mu m}$. The function $\xi_{\text{lower}}(\varphi)$ of the lower ground QE state is

$$
\xi_{\text{lower}}(\varphi) = 1 + 2 \sum_{m=1}^{2\Delta m / \mu} \cos[(\mu m - \pi) \varphi] \exp\left( -\frac{(\mu m)^2}{4a_e^2} \right). \quad (24)
$$

The maxima of the function $\xi(\varphi)$ in Eq. (23) correspond to minima of the $\xi_{\text{lower}}(\varphi)$ in Eq. (24), and vice versa. Thus, for $\mu = 1$ the function $\xi_{\text{lower}}(\varphi)$ has a maximum at $\varphi = \pi$; at $\mu = 2$ there are two maxima at $\varphi = \pm \pi/2$ and so on. In general, the Husimi functions of the two QE ground states have $2\mu$ maxima with the radius $r = r_c$. Each maximum is situated at the center of a quantum resonance cell, so that the quantum phase space has the same symmetry as the classical phase space. In Figs. 2(a) and 2(b) the Husimi functions of the extreme QE states of four cells are plotted for the same parameters as in Fig. 1 and for two values of the effective Planck constant $\hbar$. It is obvious that the symmetry of the Husimi function, shown in Figs. 2(a) and 2(b), is the same as the symmetry of the classical phase space in Fig. 1. A similar result was demonstrated numerically in Ref. 10 for $\mu = 1$. Since we made the approximation (18), the centers of the quantum cells are slightly shifted in $r$ in comparison with the positions of the stable points in the classical phase space. This difference decreases with $h$ decreasing and for the cells with larger values of $r$. As follows from a comparison of Figs. 2(a) and 2(b) the width of the Husimi function decreases when $h$ decreases. This is the result of narrowing of the relative width of the Gaussian wave packets (12) with decreasing $h$, mentioned earlier. In the quasiclassical limit, when $h$ tends to zero, the two extreme Husimi functions of one cell degenerate into $2\mu$ $\delta$ functions located at the elliptic stable points in the phase space.\textsuperscript{14,15}

In summary, we presented an analytical approach and numerical analysis, based on the Husimi function, which allowed us to analyze the characteristic global properties of the quantum phase space, in the resonance approximation. It is shown analytically that the Husimi functions of all the Floquet states in the regime of weak chaos possess the same
symmetry (axial symmetry of the order $2\mu$) as the symmetry of the classical phase space. This correspondence is demonstrated numerically for the Floquet states responsible for the motion near the elliptic stable points. It is shown that the quantum phase space is symmetric only in the resonance approximation (like in the classical case) when we neglect the exponentially thin chaotic component in the region near separatrices. The developed approach can be used for studying other quantum systems which have an infinite separatrix net in the classical limit.

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