Control of implicit chaotic maps using nonlinear approximations

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The technique of using nonlinear approximations to design controllers for chaotic dynamical systems introduced by Yagasaki and Uozumi is extended in order to enable it to be used to design controllers for chaotic dynamical systems that are described by implicit maps and is then used to control the well-known bouncing ball system without recourse to the high-bounce approximation. © 2000 American Institute of Physics. [S1054-1500(00)01803-6]

While chaotic systems appear to behave randomly, there is in fact an underlying deterministic dynamics. This can be used to aid in the design of procedures for controlling such systems, that is, making them behave in desired ways. Sometimes a system will be a model of an application in which one or more of its parameters is able to be varied, and a variety of controllers have been designed to take advantage of this situation when it arises. These usually rely on the chaotic nature of the system to, at some stage, bring the system close to the desired configuration and from there make parameter adjustments as necessary. Unfortunately, this can take a long time to happen. Recently, methods that can greatly reduce the waiting time have been developed. This problem is particularly critical if the system modeling the application is defined implicitly, and cannot be faithfully approximated explicitly, because the calculation at each new time point requires the solution of a set of nontrivial equations. In this study, we show how one of the newly developed methods can be modified so as to make it applicable to such systems.

I. INTRODUCTION

Several ways of controlling chaotic dynamical systems have been proposed over the past decade. A good recent reference to many of these is the book by Chen and Dong. In many situations the control designer has access to a system parameter which can be varied, but usually only within a certain range because of engineering and/or financial restrictions. The first method proposed to control such dynamical systems was what is now known as the OGY method, which can be used to stabilize saddle-type fixed points (and periodic orbits, in general) embedded in attractors of chaotic dynamical systems which have a variable system parameter. It consists of linearizing the original system in the neighborhood of the fixed point and applying a small perturbation to the parameter when the system approaches sufficiently close to the fixed point so as to force it onto the stable manifold of the local linear system. The ergodic nature of the chaotic dynamical system guarantees that it will eventually land in a neighborhood of the fixed point, however this may take a very long time to occur.

Various methods have been proposed to increase the effectiveness of this approach. One such method is that of Yagasaki and Uozumi, which will be referred to here as the YU method. It is a way of extending the OGY method by using nonlinear approximations of the chaotic dynamical system and the stable manifold of the fixed point in order to increase the size of the region in which the parameter variation control is activated. When the system lands in a neighborhood of the stable manifold of the target point the control parameter is perturbed in such a way as to force the system onto the stable manifold from where it will naturally move in the direction of the fixed point. The method has been shown to be useful in a variety of applications, and a way of further increasing its effectiveness has been proposed by the present author.

There are many situations in which it is necessary to employ an implicit map to model a system accurately. One such example occurs in the manufacture of anodes used in the production of aluminum. Anodes are produced from petroleum coke, recycled spent anodes, and a binding pitch. The coke and recycled material are crushed to a desired size, mixed and heated. Pitch is then added and further heating produces a paste. This paste is then formed into large blocks in a vibratory compactor, or vibroformer. The blocks are then baked to produce the anodes. The quality of the anodes has a significant effect on the efficiency of the reduction process. Homogeneous, well-compacted anodes are desired and as such, vibration compaction is much more effective than monotone compression, particularly in the elimination from the blocks of air bubbles which can cause major cracks in the anode, resulting in increased carbon consumption and possible process destabilization. Experience has shown that vibroformer motion that is too regular tends to produce low-quality anodes. However, irregular oscillations can produce too high a jump of the compactor which can result in cracking of the anode. Hence, to reduce costs and improve productivity, it is desirable to be able to control and optimize the vibroformer operation so as to make high-quality anodes. See Ref. 7 for a more detailed description of the vibroforming process and an applicable control technique.

The vibroformer can be modeled as a bouncing-ball sys-
system, the map of which is implicit. In many cases the bouncing-ball system can be simplified by assuming that the ball has a high bounce, which leads to an explicit map. However, as stated above, the vibroformer is required to not bounce very high so this simplifying assumption is invalid and we must work with the full (implicit) map.

In this article we show how the YU method can be modified so that it can be used to control dynamical systems that are described by implicit maps.

In Sec. II we briefly review the YU method. In Sec. III we modify the YU method so as to make it applicable to implicit maps. In Sec. IV we demonstrate the effectiveness of the YU method by applying it to a number of maps including the bouncing-ball map. In Sec. V we present a summary and conclusions.

II. THE YU METHOD

For simplicity we consider only dynamical systems that can be modeled using the one-parameter two-dimensional discrete map

\[ x_{n+1} = \phi(x_n, \mu_n) \]

where \( x=(x,y)^T \) is the state variable of the system and \( \mu \) is an accessible parameter of the system. We assume that for parameter values in the neighborhood of a nominal value \( \bar{\mu} \) the system has a saddle point \( x^*(\mu) \) and that chaotic motion occurs. In particular, let \( \dot{x}^* = x^*(\bar{\mu}) \). The system starts from an initial point \( x_0 \) with the nominal parameter value \( \mu \). A simple nonlinear (polynomial) approximation to the dynamical system is employed in order to determine a polynomial approximation to the stable manifold \( \dot{W}^s \) of \( \dot{x}^* \). The system is allowed to progress until it lands sufficiently close to (the approximation of) the stable manifold. When this occurs a small perturbation is applied to the accessible parameter in order to drive the system onto the stable manifold.

The linearization of the map with \( \mu = \bar{\mu} \) around its unstable fixed point is

\[ x_{n+1} = \dot{x}^* + M(x_n - \dot{x}^*), \quad M = D \phi(\dot{x}^*, \bar{\mu}). \]  

The matrix \( M \) will have eigenvalues \( \lambda_s \) and \( \lambda_u \) corresponding to the stable and unstable directions, respectively. We note that \( |\lambda_s| < 1 < |\lambda_u| \). We label the corresponding (column) eigenvectors \( e_s \) and \( e_u \).

At this stage, Yagasaki and Ouzumi introduced a coordinate transformation based on these eigenvectors. Their approach works equally well in the space of the original variables and it is in this space that we shall stay. This is particularly significant when working with periodic maps such as the bouncing ball map. Handling the periodicity situation in the space of the transformed variables is nontrivial.

We write \( \dot{x}^* = (\dot{x}^*, \dot{y}^*)^T \) and \( \phi(x, \mu) = (f(x, y, \mu), g(x, y, \mu))^T \) and assume that polynomial approximations for the components can be computed.

Making use of these we can compute a polynomial approximation to the stable manifold \( \dot{W}^s \) in the form

\[ y = h(x) = \dot{y}^* + \gamma_s(x - \dot{x}^*) + \sum_{i=2}^{\infty} h_i(x - \dot{x}^*), \]

\[ \gamma_s = \frac{e_{s2}}{e_{s1}}. \]

We need to calculate the first-order coefficient individually so as to guarantee that we find the stable manifold and not the unstable one. Since points on \( \dot{W}^s \) are mapped to points on \( \dot{W}^s \), then the equation of the stable manifold must satisfy the polynomial equation

\[ m(x, h(x), \bar{\mu}) = 0 \quad \text{where} \quad m = g - h(f), \]

from which the unknown coefficients can be determined.

The ergodicity of the system guarantees that after a number of steps the system gets sufficiently close to \( \dot{W}^s \), that is, \( |x_n - \dot{x}^*| < d_x \) and \( |y_n - h(x_n)| < d_y \), for some preassigned (small) distances \( d_x \) and \( d_y \). When this happens we calculate a new parameter value which should be a small perturbation of the nominal value. We note that at this stage \( \mu_n \) is unknown, but will be assumed to be close to \( \bar{\mu} \). If the next point is to be on \( \dot{W}^s \), then we must have \( m(x_n, y_n, \mu_n) = 0 \).

We employ a Taylor expansion to get (to first order)

\[ (y_n - h(x_n))m_s(x_n, h(x_n), \mu) + (\mu_n - \bar{\mu})m_p(x_n, h(x_n), \mu) = 0, \]

from which we deduce that

\[ \mu_n = \bar{\mu} - \frac{m_s(x_n, h(x_n), \bar{\mu})}{m_p(x_n, h(x_n), \bar{\mu})}(y_n - h(x_n)). \]

III. IMPLICIT MAPS

The above analysis can be modified to cater for situations in which the model equation is implicit, as follows. For simplicity we consider only dynamical systems that can be modeled using the implicitly defined one-parameter two-dimensional discrete map

\[ A(x_n, x_{n+1}, \mu_n) = 0, \quad B(x_n, x_{n+1}, \mu_n) = 0, \]

where \( x=(x,y)^T \) is the state variable of the system and \( \mu \) is an accessible parameter.

We assume that for parameter values in the neighborhood of a nominal value \( \bar{\mu} \) the system has a saddle point \( x^*(\mu) \) and that chaotic motion occurs. Let \( \dot{x}^* = x^*(\bar{\mu}) \). Linearization of Eqs. (7) with the nominal value \( \bar{\mu} \) around the unstable fixed point leads to a system of linear equations that can be put in the form \( \dot{x_{n+1}} = \dot{x}^* + M(x_n - \dot{x}^*) \). As before, the matrix \( M \) will have eigenvalues \( \lambda_s \) and \( \lambda_u \) corresponding to the stable and unstable directions, respectively. We label the corresponding (column) eigenvectors \( e_s \) and \( e_u \), and let \( y = e_{s2}/e_{s1} \).

We write \( \dot{x}^* = (\dot{x}^*, \dot{y}^*)^T \) and Eqs. (7) as

\[ A(x_n, y_n, x_{n+1}, y_{n+1}, \mu_n) = 0, \]

\[ B(x_n, y_n, x_{n+1}, y_{n+1}, \mu_n) = 0. \]

We shall assume polynomial approximations for these maps. If a point \( (x, y) \) is on the stable manifold, then its image \( (X, Y) \) will also be. Hence the equation of the stable manifold \( y = h(x) \) must satisfy

\[ A(x, h(x), \dot{X}(x), h(\dot{X}(x)), \bar{\mu}) = 0, \]

\[ B(x, h(x), \dot{X}(x), h(\dot{X}(x)), \bar{\mu}) = 0, \]
where \( \hat{X}(x) = X(x, h(x)) \).

Note that the explicit map \( x_{n+1} = f(x_n, y_n, \mu) \), \( y_{n+1} = g(x_n, y_n, \mu) \) is equivalent to the implicit map \( A = x_{n+1} - f(x_n, y_n, \mu) = 0 \), \( B = y_{n+1} - g(x_n, y_n, \mu) = 0 \), from which we can deduce Eq. (4) of the previous section.

Equations (9) are a pair of polynomial equations in \( x \) from which we can determine \( h(x) \) and \( \hat{X}(x) \) in the form

\[
\begin{align*}
    h(x) &= \hat{x}^* + y(x - \hat{x}^*)(1 + \sum_{i=2}^{\infty} h_i(x - \hat{x}^*)^i), \\
    \hat{X}(x) &= \hat{x}^* + x(x - \hat{x}^*)(1 + \sum_{i=2}^{\infty} X_i(x - \hat{x}^*)^i).
\end{align*}
\]

When the system gets sufficiently close to the stable manifold we calculate a new parameter value which should be a small perturbation of the nominal value. We note that at this stage \( \mu_n \) is unknown, but will be assumed to be close to \( \hat{\mu} \). If the next point is to be on \( \hat{W}^s \), then we must have

\[
\begin{align*}
    A(x_n, y_n, x_{n+1}, h(x_{n+1}), \mu_n) = 0, \\
    B(x_n, y_n, x_{n+1}, h(x_{n+1}), \mu_n) = 0.
\end{align*}
\]

Since the system is assumed to be close to \( \hat{W}^s \) then, to first order, \( x_{n+1} = \hat{X}(x_n) \) and we can use either of Eqs. (11) to find \( \mu_n \). Using the first one, we get

\[
A(x_n, y_n, \hat{X}(x_n), h(\hat{X}(x_n)), \mu_n) = 0
\]

and employing a Taylor expansion we get (to first order)

\[
\begin{align*}
    A(v_n) + (y_n - h(x_n))A_x(v_n) + (\mu_n - \hat{\mu})A_\mu(v_n) = 0, \\
    v_n = (x_n, h(x_n), \hat{X}(x_n), h(\hat{X}(x_n)), \hat{\mu})
\end{align*}
\]

and from this we calculate that

\[
\mu_n = \hat{\mu} - \frac{A_v}(v_n) + (y_n - h(x_n))A_x(v_n) \Big| \frac{A_\mu(v_n)}{A_\mu(v_n)}
\]

The modeling process may lead to situations in which Eqs. (8) will have multiple solutions. Usually, only one of these will be physically realistic and it is this consideration that must be used to choose between these or to direct the solution procedure to the correct one, as appropriate. We note that this will not be necessary, however, in situations where the controller receives its status information from measuring devices attached to the system that is being controlled because the control equation (14) only needs the current position of the system.

### IV. APPLICATIONS

In this section we demonstrate the effectiveness of the YU method by using it to stabilize unstable fixed points of a number of maps. First, by way of a simple example of the method we apply it to the well-known Hénon map. Next, we discuss the bouncing-ball application and derive its corresponding map and also that of the simplified (high-bounce approximation) system. Having done this, we use the YU method to stabilize one of the fixed points of the explicit map and then use the extended YU method to stabilize one of the fixed points of the implicit map.

#### A. The Hénon map

The Hénon map

\[
x_{n+1} = -x_n^2 + \beta y_n, \quad y_{n+1} = x_n,
\]

with \( \alpha \) as the control parameter, was used by Yagasaki and Uozumi to demonstrate their method (in the space of transformed variables). We shall use \( \beta = 0.3 \) and \( \alpha = 1.4 \), and all calculations are done in MAPLE 5, Release 5.1, on a Toshiba Satellite 2510CDS.

The above map has an unstable fixed point at approximately \( \hat{x}^* = (0.8839, 0.8839) \) which lies in its strange attractor. Figure 1 illustrates 500 iterations of the map starting from \((x_0, y_0) = (0.5, -0.5)\) along with the target point and the quadratic approximation to its stable manifold.

#### B. The bouncing-ball system

The bouncing-ball system consists of a ball bouncing on an oscillating table. Its map is implicit, but usually the so-called high-bounce approximation is assumed, which has the...
advantage of making the map explicit. This map is very well understood.\textsuperscript{8,9} Unfortunately, this approximation is invalid if it does not accurately reflect the application that is being modeled. This is indeed the case in the vibroformer application.

We take the motion of the table to be given by \( w(t) = a \sin \omega t \) where \( w(t) \) is the displacement of the table in the vertical direction, \( a \) is the amplitude of its vibration, \( \omega \) is the frequency of its vibration, and \( t \) is time. We let \( z(t) \) denote the displacement of the ball. Obviously, \( z(t) = w(t) \) at times of impact. Between such times \( w(t) \) will follow the above sinusoidal law and \( z(t) \) will follow the usual gravity law.

We let \( t_n \) denote the time of the \( n \)th impact, \( v_n \) the velocity with which the ball hits the table at the \( n \)th impact and \( v_n^+ \) denote the velocity with which the ball leaves the table after the \( n \)th impact. Then by conservation of momentum and the definition of the coefficient of restitution we have

\[
v_n^+ = \left( \frac{v_n - \beta}{1 + \nu} \right) v_n^- + \frac{1 + \beta}{1 + \nu} a \omega \cos \omega t_n, \tag{17}\]

where \( \nu \) is the mass ratio of the ball to the plate and \( \beta \) is the coefficient of restitution.

In the case of the high-bounce approximation, we assume that \( \alpha \) is negligible compared with the height of the bounce, in which case we have that from one bounce to the next

\[
v_{n+1}^- = -v_n^+, \quad t_{n+1} = t_n + \frac{2}{g} v_n^+, \tag{18}\]

where \( g \) is the acceleration due to gravity. Combining Eq. (17) evaluated at \( n+1 \) and Eq. (18), and writing \( v_n = v_n^+ \) we get

\[
v_{n+1} = a v_n + a \omega (1 + \alpha) \cos \omega t_{n+1}, \quad \alpha = \frac{\beta - \nu}{1 + \nu}. \tag{19}\]

In total, we have the two-dimensional explicit map

\[
t_{n+1} = t_n + \frac{2}{g} v_n, \tag{20}\]
\[
v_{n+1} = a v_n + a \omega (1 + \alpha) \cos \left( \omega t_n + \frac{2}{g} v_n \right) . \tag{21}\]

If the high-bounce approximation is not used, then given the time of the \( n \)th impact, the time of the next impact is the next instant when the ball and the table are at the same height. For \( t_n < t < t_{n+1} \) the ball follows the curve

\[
z(t) = -\frac{g}{2} (t - t_n)^2 + v_n (t - t_n) + a \sin \omega t_n. \tag{22}\]

Hence the time of the next impact is obtained by solving the implicit equation

\[
-\frac{g}{2} (t_{n+1} - t_n)^2 + v_n (t_{n+1} - t_n) + a \sin \omega t_n = a \sin \omega t_{n+1} . \tag{23}\]

Using \( v^{-1}(t) = z(t) \) in Eq. (17) evaluated at \( n+1 \) and writing \( v_n = v_n^+ \) we obtain

\[
v_{n+1} = -a (v_n - g (t_{n+1} - t_n)) + a \omega (1 + \alpha) \cos \omega t_{n+1} . \tag{24}\]

These last two equations define the two-dimensional implicit map. Both maps can be nondimensionalized by introducing new variables

\[
\bar{t} = \omega t, \quad \bar{v} = \frac{2 \omega}{g} v, \quad \mu = \frac{2a \omega^2 (1 + \alpha)}{g} . \tag{25}\]

In terms of these variables, the explicit map is

\[
\bar{t}_{n+1} = \bar{t}_n + \bar{v}_n, \quad \bar{v}_{n+1} = a \bar{v}_n + \mu \cos \bar{t}_n, \tag{26}\]

and the implicit map is

\[
-\frac{\mu}{1 + \alpha} \sin \bar{t}_n + \frac{\mu}{1 + \alpha} \sin \bar{t}_{n+1} = 0, \tag{27}\]

\[
\bar{v}_{n+1} + a (\bar{v}_n - 2 (\bar{t}_{n+1} - \bar{t}_n)) - \mu \cos \bar{t}_{n+1} = 0. \tag{28}\]

Under the explicit map (20) negative values for velocity are possible (corresponding to time reversal or negative bounce height). In order to correct this, whenever the map gives a negative value for \( v_{n+1} \) it is replaced by \( v_{n+1} = -a \omega (1 + \alpha) \cos \omega t_n \) which is the negative of the velocity with which the ball would have left the plate had it hit with zero velocity. Also, in order to obtain equilibrium solutions from this system the time variable must be evaluated modulo \( 2\pi \omega / a \), making it effectively a phase variable. Hence, the complete map, in nondimensional form, is

\[
\bar{t}_{n+1} = \bar{t}_n + \bar{v}_n (\text{mod} \ 2\pi), \tag{29}\]
\[
\bar{v}_{n+1} = \alpha \bar{v}_n + \mu \cos (\bar{t}_n + \bar{v}_n), \tag{30}\]

and has fixed points \((\tilde{\bar{t}}^*, \tilde{\bar{v}}^*)\) where
\hat{t}^* = 2\pi, \quad \frac{2\pi(1-\alpha)}{\mu} = \cos \hat{t}^*.

There are two such points if \( \mu > 2\pi(1-\alpha) > 0 \), which we shall assume to be the case. The stability of the point depends on the parameter values chosen. Vincent\(^{10}\) used a Lyapunov function approach to direct the system onto a stable manifold. We shall use the YU method to stabilize an unstable fixed point assuming that we have access to the parameter \( \mu \) through variation in \( \omega \). In particular we shall choose the one with the lesser of the two \( \hat{t}^* \) values. The stable eigenvalue and corresponding eigenvector gradient are

\[ \lambda_2 = \frac{c + \sqrt{c^2 - 4\alpha}}{2}, \quad c = 1 + \alpha - \hat{\mu} \sin \hat{t}^*, \quad \gamma_2 = \lambda_2 - 1. \]

We choose parameter values \( \alpha = 0.8, \hat{a} = 0.01, g = 9.8, \) and \( \omega = 50 \), so \( \hat{\mu} = 9.184 \) and the fixed point is a saddle. In particular,

\[ \hat{t}^* = 1.4335, \quad \hat{\theta}^* = 6.2832, \quad \lambda_2 = -0.1113, \]

\[ \lambda_3 = -7.1860, \quad \gamma_3 = -1.1113. \]

Figure 3 illustrates 500 iterations of the map for these parameter values, starting from \((\hat{t}_0, \hat{\theta}_0) = (0, \hat{\mu})\). Also shown is the target point and the quadratic approximation to its stable manifold \( \bar{\theta} = h(\bar{t}) \). We note that the stable manifold is very nearly linear. The parabolic arc is an artifact of the negative velocity modification mentioned earlier.

The YU method is quite robust and in fact can often achieve very good results using low-order approximations of the stable manifold and the ratio appearing in Eq. (6), which determines the control parameter variation.\(^5\) Figure 4 depicts the stabilization of the unstable fixed point \((\hat{t}^*, \hat{\theta}^*)\) = \((1.4335, 6.2832)\), starting from \((\hat{t}_0, \hat{\theta}_0) = (0, \hat{\mu})\), using quadratic approximations for \( h(\hat{t}) \) and \( \hat{t}(\hat{t}) \) and a linear approximation for the ratio, and proximity parameters \( d\bar{t} = 1 \) and \( d\bar{\theta} = 0.5102 \) (corresponding to \( dt = 0.02 \) and \( d\theta = 0.05 \)). Stabilization was achieved in 88 iterations and the maximum parameter variation was 3.50%.

The implicit map does not suffer from the problem of unrealistic velocities. They can go negative but are never more negative than that of the table on impact. However, as for the explicit map, in order to obtain equilibrium solutions from the system the time variable must be evaluated modulo \( 2\pi/\omega \). Hence, the complete map, in nondimensional form, is

\[ - (\hat{t}_{n+1}^\dagger - \hat{t}_n)^2 + \bar{v}_n(\hat{t}_{n+1}^\dagger - \hat{t}_n) + \frac{\mu}{1 + \alpha} \sin \hat{t}_n^\dagger \]

\[ - \frac{\mu}{1 + \alpha} \sin \hat{t}_{n+1} = 0, \]

\[ \bar{v}_{n+1} + \alpha(\bar{v}_n - 2(\hat{t}_{n+1}^\dagger - \hat{t}_n)) - \mu \cos \hat{t}_{n+1} = 0, \]

and also has the fixed points (28). We note that Eqs. (31) can have multiple solutions, but only one of these will be realistic. It is that one which corresponds to the time of next contact of the ball with the table. An analysis of the trigonometric structure of these equations can be used to design a simple forward-search algorithm.

We shall use the modified YU method to stabilize the unstable fixed point with lesser \( \hat{t}^* \) value assuming that we have access to the parameter \( \mu \) through variation in \( \omega \). The stable eigenvalue and corresponding eigenvector gradient are

\[ \lambda_2 = \frac{c + \sqrt{c^2 - 4\alpha^2}}{2}, \quad c = 1 + \alpha^2 - \hat{\mu}(1 + \alpha) \sin \hat{t}^*, \]

\[ \gamma_2 = \frac{2(\lambda_2 - 1)}{1 + \alpha}. \]

We again choose parameter values \( \alpha = 0.8, \hat{a} = 0.01, g = 9.8, \) and \( \omega = 50 \), so \( \hat{\mu} = 9.184 \) and the fixed point is a saddle. In particular,
The stable manifold is the target point and the tenth-order approximation to its quadratic approximations are sufficient given the values of the control parameter variation formula derived using the velocity equation, and proximity parameters \(d\bar{t} = 1\) and \(d\bar{y} = 0.5102\) (corresponding to \(dt = 0.02\) and \(dy = 0.05\)). The quadratic approximations are sufficient given the values of \(\hat{t}^*\) and \(\bar{t}^*\). Stabilization was achieved in 61 iterations and the maximum parameter variation was 7.17%.

The stabilization time is shorter than that for the explicit map because of the greater concentration of system points around the fixed point, as can be seen by comparing Figs. 3 and 5.

Maximum parameter variation can generally be reduced by tightening of the proximity parameters although this leads to an increase in the stabilization waiting time. When the time proximity parameter was changed to \(d\bar{t} = 0.5\) (corresponding to \(dt = 0.01\)) the system was stabilized in 66 iterations and the maximum parameter variation was reduced to 3.58%.

V. CONCLUSION

We have shown how the YU method of using nonlinear approximations to design controllers for chaotic dynamical systems that have an accessible parameter can be modified to cater for situations in which the map describing the chaotic dynamical system is implicit.

Since most of the analysis is carried out before hand, the YU method creates controllers that are very simple to implement and also produce rapid results because once activated, control is achieved in only a handful of iterations.

In some cases the amount of allowable parameter variation can be built into the controller. Stabilization of one of the unstable fixed points of the Hénon map was achieved in only 19 iterations and with only 2% parameter variation.

The bouncing-ball map is a much more elaborate map but control was still achieved with very little effort. The explicit map only required substitution into a linear expression to calculate the parameter variation and the implicit map only required substitution into a quadratic expression. Again, once activated, control was quickly achieved.

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