Mono- and multi-fractal investigation of scaling properties in temporal patterns of seismic sequences

Luciano Telesca a,*, Vincenzo Lapenna a, Maria Macchiato b

a Institute of Methodologies for Environmental Analysis, National Research Council, Tito Scalo, PZ, Italy
b Dipartimento di Scienze Fisiche, Università “Federico II”, Naples, Italy

Accepted 24 April 2003

Communicated by Prof. I. Antoniou

Abstract

Understanding the statistical properties of time-occurrence series of seismic sequences is considered one of the most pervasive scientific topics. Investigating into the patterns of seismic sequences reveals evidence of time-scaling features. This is shown in the fractal analysis of the 1986–2001 seismicity of three different seismic zones in Italy. Describing the sequence of earthquakes by means of the series of the interevent times, power-law behaviour has been found applying Hurst analysis and detrended fluctuation analysis (DFA), with consistent values for the scaling exponents. The multifractal analysis has clearly evidenced differences among the earthquake sequences. The multifractal spectrum parameters (maximum $a_0$, asymmetry $B$ and width $W$), derived from the analysis of the shape of the singularity spectrum, have been used to measure the complexity of seismicity.

© 2003 Elsevier Ltd. All rights reserved.

1. Introduction

Much recent work has been focused on the understanding the dynamical variability of the most of the geophysical phenomena. A wide variety of natural phenomena, as examples of scale invariant sets, suggests that many complex processes may be interpreted by means of concepts of scale-invariance, fractals and multifractals, used to better describe the feature of inhomogeneity, which characterises that variability. The existence of power-laws (scaling) in the statistics used to describe the patterns of temporally fluctuating systems indicates the presence of a fractal behaviour.

In this context, the analysis of time-scaling properties of seismic processes can give useful information on the dynamical features of seismic processes and on the involved geodynamical mechanisms. Understanding of the physical mechanism underlying the presence of scaling laws in earthquake dynamics is a very fascinating research topic. The appearance of time self-similar behaviour of seismicity is a measure of the degree of both the heterogeneity of the process and the clustering of seismic activity. Many studies based on earthquake catalogues have demonstrated the presence of clustering in the temporal distribution of earthquakes (e.g. [17]). The analysis of the temporal variations of the scaling properties of earthquakes is a powerful tool to characterise the main features of seismicity and to bring us insight the inner dynamics of seismotectonic activity [32].

Scale-invariance characterises earthquake dynamics in different features. Gutenberg and Richter [12] found that earthquake size follows a power-law distribution. Fractal and multifractal tools applied to seismicity have been used in
the pioneer work of Smalley et al. [28]. Scale-invariant behaviours were determined in Kagan and Jackson [16]; Kagan [15] reviewed experimental evidences for earthquake scale-invariance. A theory to explain the presence of scale-invariance in earthquakes was proposed by Bak and Tang [1], by introducing the concept of self-organized criticality (SOC). The analysis of scaling laws concerning earthquakes has led to the development of a wide variety of physical models of seismogenesis [29,30], used to better characterise the seismicity patterns [1,5,6,14,29,30]. Several distributions have been used to model seismic activity. Among these, the Poisson distribution, which implies the independence of each event from the time elapsed since the previous event, is the most extensively used, since, in many cases, for large events a simple discrete Poisson distribution provides a close fit [3]. However, in recent studies it has been shown that earthquake occurrence is characterised by temporal clustering properties with both short and long timescales [16], this implying the presence of time-correlation among the seismic events. The presence of time correlations in the temporal distribution of a seismic series has been performed in Bittner et al. [2] and Lapenna et al. [19] by the estimate of the power spectral density by means of the variance–time curve (VTC) method, that for spectra with power-law (fractal) shape furnishes a relation with the spectral exponent \( \alpha \). Recently, the estimate of the scaling exponent of the power spectral density of an earthquake sequence has been performed using two statistics, the Fano Factor and the Allan Factor, that have a power-law behaviour for processes with scaling properties [32]. The use of ranged scale analysis (R/S) to estimate the Hurst exponent of the interevent time series has been carried out in Correig et al. [7], revealing a persistence character of seismicity. All these measures are consistent with each other [33], so that we can define one scaling exponent, that is sufficient to capture the dynamics of a seismic process. But one scaling exponent can completely describe a monofractal. Monofractals are homogeneous in the sense that they have the same scaling properties, characterised by a single singularity exponent ([31], and references therein). The need for more than one scaling exponent to describe the scaling properties of the process uniquely, indicates that the process is not a monofractal but could be a multifractal. A multifractal object requires many indices to characterise its scaling properties. Multifractals can be decomposed into many—possibly infinitely many—subsets characterised by different scaling exponents. Thus multifractals are intrinsically more complex and inhomogeneous than monofractals [31], and characterise systems featured by a spiky dynamics, with sudden and intense bursts of high frequency activity [9,10], qualified by the idea of intermittency (e.g. [4,23,25]), characterizing many complex physical systems in nature for which energy at a given timescale is not evenly distributed in time [21].

A seismic process can be considered as characterised by a fluctuating behaviour, that is temporal phases of low activity are interspersed between those where the density of the events is relatively large. This “sparseness” can be described by means of the concept of multifractal [9,10]. The multifractal method of analysis is an efficient tool to describe the intermittent fluctuations of geophysical and biological systems [24], and it seems well suited to investigate the variability of the earthquake activity.

Maybe the most adequate manner to investigate multifractals is to analyze their fractality or singularity spectra. The singularity spectrum quantifies the fractal dimension of the subset characterised by a particular exponent, that is gives information about the relative dominance of various fractal exponents present in the process. In particular, the maximum of the spectra furnishes the dominant fractal exponent and the width of the spectrum denotes the range of the fractal exponents.

In this paper we analyze the earthquake sequences from 1986 to 2001 recorded in three different seismic zones in Italy (Fig. 1) by means of monofractal and multifractal methods, in order to qualitatively and quantitatively characterise the fluctuating dynamics of seismicity.

2. Methods

A seismic sequence can be represented by a temporal point process, that describes events that occur at some random locations in time [8], and it can be expressed by a finite sum of Dirac’s delta functions centered on the occurrence times \( t_i \), with amplitude \( A_i \) proportional to the magnitude of the earthquake:

\[
y(t) = \sum_{i=1}^{N+1} A_i \delta(t - t_i),
\]

(1)

\((N + 1)\) represents the number of events recorded.

This process can be described by the set of the interevent times, that is the series of the time intervals between two successive earthquakes.

Therefore, we can produce a discrete time series \( \{ \tau_i \} \)

\[
\tau_i = t_{i+1} - t_i.
\]

(2)

with the subscript \( i \) ranging from 1 to \( N \).
2.1. R/S analysis

The Hurst exponent [13] provides information about correlations among blocks of interevent intervals. Being \( \tau_i \) the interevent interval and \( i \) the event number with \( 1 \leq i \leq N \), define the average interevent time over the period \( m \) as

\[
\langle \tau \rangle_m = \frac{\sum_{i=1}^{m} \tau_i}{m}.
\]

(3)

Define the accumulated departure \( X(i) \) of \( \tau_i \) from the mean \( \langle \tau \rangle_m \) as

\[
X(i, m) = \sum_{u=1}^{i} (\tau_u - \langle \tau \rangle_m).
\]

(4)

The range \( R(m) \) is the difference between the minimum and the maximum accumulated departure \( X \)

\[
R(m) = \max_{1 \leq i \leq m} X(i, m) - \min_{1 \leq i \leq m} X(i, m)
\]

(5)

with \( m \) ranging from 2 to \( N \).

The standard deviation \( S \) over the period \( m \) is

\[
S(m) = \sqrt{\frac{\sum_{i=1}^{m} (\tau_i - \langle \tau \rangle_m)^2}{m}}.
\]

(6)

The rescaled range is the range of the deviations rescaled or renormalized by the standard deviation

\[
\frac{R(m)}{S(m)} \propto m^H
\]

(7)

with \( H \) the Hurst exponent. The deviation of the cumulative interevent time from its average trend can provide us information about the cycles of short or larger intervals of time for the occurrence of the next event [7]. For a purely
random process, with no correlations among intervals, $H = 0.5$. For other stochastic processes, $H$ can be less or more large than 0.5; for $H > 0.5$ the process is characterised by a persistent behaviour, meaning that short (long) interevent times are more likely to be followed by short (long) ones; for $H < 0.5$ the process is characterised by antipersistent behaviour, meaning that short (long) interevent times are more likely to be followed by long (short) ones. This measure generally describes processes that are characterised by long-term correlations, providing a good index of correlations due to the ordering of the interevent intervals alone.

2.2. Detrended fluctuation analysis

The DFA was proposed by Peng et al. [22], and it avoids spurious detection of correlations that are artifacts of nonstationarity, that often affects experimental data. Such trends have to be well distinguished from the intrinsic fluctuations of the system in order to find the correct scaling behaviour of the fluctuations. Very often we do not know the reasons for underlying trends in collected data and we do not know the scales of underlying trends. DFA is a method for determining the scaling behaviour of data in the presence of possible trends without knowing their origin and shape [18].

The methodology operates on the time series $\tau_i$, where $i = 1, 2, \ldots, N$ and $N$ is the length of the series. With $\bar{\tau}$ we indicate the mean interevent time. The series is first integrated

$$y(k) = \sum_{i=1}^{k} [\tau_i - \bar{\tau}]$$

with $k = 1 \ldots N$.

Next, the integrated time series is divided into boxes of equal length, $n$. In each box a least-squares line is fit to the data, representing the trend in that box. The $y$ coordinate of the straight line segments is denoted by $y_n(k)$. Next we detrend the integrated time series $y(k)$ by subtracting the local trend $y_n(k)$ in each box. The root-mean-square fluctuation of this integrated and detrended time series is calculated by

$$F(n) = \sqrt{\frac{1}{N} \sum_{k=1}^{N} [y(k) - y_n(k)]^2}.$$  

(9)

Repeating this calculation over all box sizes, we obtain a relationship between $F(n)$, that represents the average fluctuation as a function of box size, and the box size $n$. If $F(n)$ behaves as a power-law function of $n$, data present scaling:

$$F(n) \propto n^d.$$  

(10)

Under these conditions the fluctuations can be described by the scaling exponent $d$, representing the slope of the line fitting $\log F(n)$ to $\log n$. For a white noise process, $d = 0.5$. If there are only short-range correlations, the initial slope may be different from 0.5 but will approach 0.5 for large window sizes. $0.5 < d < 1.0$ indicates the presence of persistent long-range correlations, meaning that a large (compared to the average) value is more likely to be followed by large value and vice versa. $0 < d < 0.5$ indicates the presence of antipersistent long-range correlations, meaning that a large (compared to the average) value is more likely to be followed by small value and vice versa. $d = 1$ indicates flicker-noise dynamics, typical of systems in a self-organized critical state. $d = 1.5$ characterises processes with Brownian-like dynamics.

2.3. Multifractal formalism

The concept of multifractal object has been introduced by Mandelbrot [20] to investigate several features in the intermittency of turbulence [21]. Many authors have developed the multifractality and its formalism, applying it to several fields of the scientific research.

Firstly, the so-called partition function $Z(q, \varepsilon)$ has to be defined,

$$Z(q, \varepsilon) = \sum_{i=1}^{N_{\text{boxes}}(\varepsilon)} \mu_i(\varepsilon).$$

(11)

The quantity $\mu_i(\varepsilon)$ is a measure and it depends on $\varepsilon$, the size or scale of the boxes used to cover the sample. The boxes are labeled by the index $i$ and $N_{\text{boxes}}(\varepsilon)$ indicates the number of boxes of size $\varepsilon$ needed to cover the sample. The exponent
q is a real parameter, that indicates the order of the moment of the measure. The choice of the functional form of the measure \( \mu_j(\varepsilon) \) is arbitrary, provided that the most restrictive condition \( \mu_j(\varepsilon) \geq 0 \) is satisfied. In the case presented in this paper, we use the following definition of measure:

\[
\mu_j(\varepsilon) = \frac{1}{T} \sum_{i=1}^{N_{box}} \tau_i,
\]

where \( \tau_i \) is the \( i \)th interevent time inside the box \( i \). \( T \) is the sum of all interevent times, and, therefore, it performs as a normalization constant, so that \( \sum \mu_j(\varepsilon) = 1 \); with this normalization, the measure \( \mu_j(\varepsilon) \) is interpreted as probability.

The parameter \( q \) can be considered as a powerful microscope, able to enhance the smallest differences of two very similar maps [11]. Furthermore, \( q \) represents a selective parameter: high values of \( q \) enhance boxes with relatively high values for \( \mu_j(\varepsilon) \); while low values of \( q \) favor boxes with relatively low values of \( \mu_j(\varepsilon) \). The box size \( \varepsilon \) can be considered as a filter, so that big values of the size is equivalent to apply a large scale filter to the map. Changing the size \( \varepsilon \), one explores the sample at different scales. Therefore, the partition function \( Z(q, \varepsilon) \) furnishes information at different scales and moments.

The generalized dimension are defined by the following equation

\[
D(q) = \lim_{\varepsilon \to 0} \frac{1}{q - 1} \ln \frac{Z(q, \varepsilon)}{\ln \varepsilon},
\]

\( D(0) \) is the capacity dimension; \( D(1) \) is the information dimension, and \( D(2) \) is the correlation dimension. An object is called monofractal if \( D(q) \) is constant for all values of \( q \), otherwise is called multifractal. In most practical applications the limit in Eq. (13) cannot be calculated, because we do not have information at small scales, or because below a minimum physical length no scaling can exist at all [11]. Generally, a scaling region is found, where a power-law can be fitted to the partition function, which in that scaling range behaves as

\[
Z(q, \varepsilon) \propto \varepsilon^{D(q)}.
\]

The slope \( \tau(q) \) is related to the generalized dimension by the following equation:

\[
\tau(q) = (q - 1)D(q).
\]

An usual measure in characterizing multifractals is given by the singularity spectrum or Legendre spectrum \( f(x) \), that is defined as follows. If for a certain box \( j \) the measure scales as

\[
\mu_j(\varepsilon) \propto \varepsilon^x
\]

the exponent \( x \), which depends upon the box \( j \), is called Hölder exponent. If all boxes have the same scaling with the same exponent \( x \), the the sample is monofractal. The multifractal is given if different boxes scale with different exponents \( x \), corresponding to different strength of the measure. Denoting as \( S_x \) the subset formed by the boxes with the same value of \( x \), and indicating as \( N_x(\varepsilon) \) the cardinality of \( S_x \), for a multifractal the following relation holds:

\[
N_x(\varepsilon) \propto \varepsilon^{-f(x)}.
\]

By means of the Legendre transform the quantities \( x \) and \( f(x) \) can be related with \( q \) and \( \tau(q) \):

\[
x(q) = \frac{dx(q)}{dq},
\]

\[
f(x) = qx(q) - \tau(q).
\]

The curve \( f(x) \) is a single-humped function for a multifractal, while reduces to a point for a monofractal.

To be able to make more quantitative statements concerning possible differences in Legendre spectra stemming from different signals, it is possible to fit, by a least square method, the spectra to a quadratic function around the position of their maxima at \( x_0 \) [26]:

\[
f(x) = A(x - x_0)^2 + B(x - x_0) + C.
\]

Parameter \( B \) measures the asymmetry of the curve, which is zero for symmetric shapes, positive or negative for left-skewed or right-skewed shapes respectively.

Another parameter is the width of the spectrum, that estimates the range of \( x \) where \( f(x) > 0 \), obtained extrapolating the fitted curve to zero; thus the width is defined as
Fig. 2. Inter-event time series for Irpinia (a), Friuli (b) and Marche (c) areas.
\[ W = \alpha_{\text{max}} - \alpha_{\text{min}} \] (21)

where \( f(\alpha_{\text{max}}) = f(\alpha_{\text{min}}) = 0 \).

These three parameters serve to describe the complexity of the signal. If \( \alpha_0 \) is low, the signal is correlated and the underlying process “loses fine structure”, becoming more regular in appearance [26]. The width \( W \) measures the length of the range of fractal exponents in the signal; therefore, the wider the range, the “richer” the signal in structure. The asymmetry parameter \( B \) informs about the dominance of low or high fractal exponents respect to the other. A right-skewed spectrum denotes relatively strongly weighted high fractal exponents, corresponding to fine structures, and low ones (more smooth-looking) for left-skewed spectra.

3. Results

After performing the Gutenberg-Richter analysis, we selected the events with magnitude \( M \geq 2.4 \), being 2.4 the minimum completeness magnitude [27]. Fig. 2 shows the interevent time series of the seismic sequences for the three seismic areas in Italy, for events with \( M \geq 2.4 \). All the sequences show a spiky dynamics, characterised by periods of high seismic activity (low interevent times) interspersed by phases of low activity (high interevent times). We performed

Fig. 3. R/S analysis of the interevent time series corresponding to sequences of earthquakes with magnitude \( M \geq 2.4 \). For all the three seismic areas the R/S curves are quite similar, with close Hurst exponents.

Fig. 4. Results of the Detrended Fluctuation Analysis applied to the time series plotted in Fig. 2. The fluctuation curves are very similar, giving very close values for the scaling coefficients.
the R/S analysis, in order to calculate the Hurst exponent. Fig. 3 shows the R/S curves for the three seismic sequences: we observe very similar scaling behaviour for all the series; the Hurst exponents are close to each other, giving \( H \approx 0.83 \) for the Irpinia series, \( H \approx 0.86 \) for the Friuli sequence and \( H \approx 0.68 \) for the Marche series. Although all the sequences are characterised by persistent fluctuations, the Irpinia and the Friuli seismic interevent time series display the most similarity. Fig. 4 shows the results of DFA. The scaling behaviours are very similar to each other with very close values for the scaling exponent \( d \): \( \approx 0.79 \) (Irpinia), \( \approx 0.80 \) (Friuli) and \( \approx 0.78 \) (Marche). In order to evaluate the variation of the scaling exponents with the threshold magnitude, we estimated the Hurst exponent (Fig. 5) and the DFA scaling exponent (Fig. 6) for the interevent time series corresponding to sequences of earthquakes with magnitude \( M \geq M_{th} \), with \( M_{th} \) ranging between 2.4 and 3.0. The variations of \( H \) with the magnitude for the Irpinia and Friuli sequences are very close to each other, clearly different from the variation of \( H \) for the Marche series. The variation of the DFA scaling exponent performs quite similarly for all the sequences, except at higher magnitudes, where the number of events plays an important role in giving the differences in the estimate of the scaling coefficient.

The differences, slightly outlined by use of the monofractal methods (Hurst analysis and DFA), among the seismic sequences are better identified by means of the multifractal method. We calculated the partition function for \( q \) ranging from \(-4.0\) to \(10.0\) with a step of \(0.5\), shown in Fig. 7a (Irpinia), b (Friuli) and c (Marche). The three seismic sequences

![Fig. 5. Variation with the threshold magnitude of the Hurst exponent. We observe a clear divergence of the Marche series from Irpinia and Friuli series, that show nearer behaviours.](image1)

![Fig. 6. Variation with the threshold magnitude of the DFA scaling exponent \( d \). The three series display very close behaviours, except at higher magnitudes, where the different number of events plays an important role in differencing the curves.](image2)
present different scaling ranges: about 1 decade for the Irpinia series, and approximately two decades for the Friuli and Marche series. To estimate the slope of the scaling region, we excluded the regions presenting the spikes at high values.

Fig. 7. Partition functions for Irpinia (a), Friuli (b) and Marche (c) varying the $q$-parameter. Linear scaling regions, from 1 to 2 decades wide, are clearly visible varying both the $q$-parameter and the box size $\epsilon$. 
of the box size $\varepsilon$. The calculation of the singularity spectrum has been performed estimating the slope $\tau(q)$ in the 1-decade linear range, common to all the sequences and to all the partition functions. The results are shown in Fig. 8; we can clearly recognize that Irpinia and Friuli catalogues present very similar spectra, different in shape from the Marche.

Fig. 8. Multifractal spectra for Irpinia (a), Friuli (b) and Marche (c) time series. All the spectra show the single-humped shape, typical of multifractal signals. By inspection the difference in shape between Marche and Irpinia, Friuli spectra is clear.
singularity spectrum. Figs. 9–11 show the variation of the multifractal parameters (maximum $a_0$, asymmetry $B$ and width $W$) with the threshold magnitude. Defining a point with the vector $(a_0, B, W)$, we constructed a 3D plot, in which each point is representative of a seismic sequence with a particular threshold magnitude (Fig. 12). We can observe that the points representative of the Marche series (solid circles) are well separated from the points representing the other two sequences. We projected the points on the planes $(a_0, B)$ (Fig. 13), $(a_0, W)$ (Fig. 14) and $(B, W)$ (Fig. 15): on the last two projections, this discrimination appears very clear.

Fig. 9. Variation with the threshold magnitude of the maximum $a_0$ of multifractal spectra.

Fig. 10. Variation with the threshold magnitude of the asymmetry $B$ of multifractal spectra.
4. Conclusions

In the present paper we performed a statistical analysis of three Italian earthquake sequences, based on monofractal methods (R/S analysis and DFA) and multifractal formalism. The R/S analysis and the DFA have revealed the presence of scaling behaviour in all the interevent time series. Persistent features have been shown by means of Hurst analysis in all the seismic sequences, also varying the threshold magnitude. Marche sequence behaves quite differently from Irpinia and Friuli sequences, assuming different values of the Hurst exponent with increasing threshold magnitude. Long-range dependence, revealed by DFA, is shown in all the sequences increasing the threshold magnitude, with very close values of the scaling exponent $d$. The multifractal analysis has given quantitative information about the “complexity” of the seismic series, revealing also the effect of an heterogeneous lithosphere, in which the heterogeneity occurs at many time scales. The determination of the multifractal parameters has been performed by means of the calculation of the Legendre spectrum. We derived three parameters, the maximum $a_0$ of the spectrum, the asymmetry $B$ and the width $W$ of the curve. This set of multifractal parameters seem to well discriminate Marche seismicity from those of Irpinia and Friuli, whose behaviours appears very similar varying the threshold magnitude of the events.

**Fig. 11.** Variation with the threshold magnitude of the width $W$ of multifractal spectra.

**Fig. 12.** 3D plot of the Irpinia (solid squares), Friuli (open circles) and Marche (solid circles) series with varying threshold magnitude in the space $(a_0, B, W)$.

4. Conclusions

In the present paper we performed a statistical analysis of three Italian earthquake sequences, based on monofractal methods (R/S analysis and DFA) and multifractal formalism. The R/S analysis and the DFA have revealed the presence of scaling behaviour in all the interevent time series. Persistent features have been shown by means of Hurst analysis in all the seismic sequences, also varying the threshold magnitude. Marche sequence behaves quite differently from Irpinia and Friuli sequences, assuming different values of the Hurst exponent with increasing threshold magnitude. Long-range dependence, revealed by DFA, is shown in all the sequences increasing the threshold magnitude, with very close values of the scaling exponent $d$. The multifractal analysis has given quantitative information about the “complexity” of the seismic series, revealing also the effect of an heterogeneous lithosphere, in which the heterogeneity occurs at many time scales. The determination of the multifractal parameters has been performed by means of the calculation of the Legendre spectrum. We derived three parameters, the maximum $a_0$ of the spectrum, the asymmetry $B$ and the width $W$ of the curve. This set of multifractal parameters seem to well discriminate Marche seismicity from those of Irpinia and Friuli, whose behaviours appears very similar varying the threshold magnitude of the events.
Fig. 13. Projection of the 3D plot in Fig. 12 on the plane $(\alpha_0, B)$.

Fig. 14. Projection of the 3D plot in Fig. 12 on the plane $(\alpha_0, W)$. 
References


Fig. 15. Projection of the 3D plot in Fig. 12 on the plane $(W, B)$. 


