Motions of curves in similarity geometries and Burgers-mKdV hierarchies

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Abstract

Integrable systems satisfied by the curvatures of curves under inextensible motions in similarity geometries are identified. It is shown that motions of curves in two-, three- and n-dimensional (n > 3) similarity geometries are described respectively by the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalizations of these hierarchies.

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1. Introduction

Many 1 + 1-dimensional integrable equations have been shown to be related to motions of inextensible curves, such relationship provides new insight and geometric explanations to properties and structure of integrable equations. In an intriguing paper of Goldstein and Petrich [1], they showed that the mKdV hierarchy arises naturally (“naturally” means that the recursion operator of the mKdV equation is appeared in the equation for the curvature) from a local motion of non-stretching plane curves in Euclidean space \( \mathbb{R}^2 \). After that, Nakayama et al. [2] obtained the sine-Gordon equation by considering a non-local motion of curves in \( \mathbb{R}^2 \), they also pointed out that the Frenet–Serret equations for curves in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) formulas are equivalent to the AKNS spectral problem without spectral parameter [2,3]. In the earlier works of Hasimoto [4], he showed that the Schrödinger equation arises from motion of inextensible curves in \( \mathbb{R}^3 \), the Schrödinger hierarchy was also obtained by Langer and Perline [7], they provided a geometrical explanation to the recursion operator of the Schrödinger equation. Using the Hasimoto transformation, Lamb [5] obtained the mKdV and sine-Gordon equations from the motion of curves in \( \mathbb{R}^3 \). The Heisenberg spin chain model which is gauge equivalent to the Schrödinger equation was derived by Lakshmanan [6]. Recently, Schief and Rogers [8] obtained an extended Harry–Dym equation and sine-Gordon equation from binormal motions of curves with constant curvature or torsion. An important trend on this topic is to study motions of curves in classical geometries. Nakayama [9] showed that the defocusing non-linear Schrödinger equation, the Regge–Lund equation, a coupled system of KdV equations and their hyperbolic type arise from motions of curves in hyperboloids in the Minkowski space. In [10] he realized the full AKNS scheme in a hyperboloid in \( M^{1,1} \). Motion of plane curves in the Minkowski space \( M^{2,1} \) was also considered by Gürses [11]. Recently we found that many 1 + 1-dimensional integrable equations including KdV, Sawada–Kotera, Burgers, Harry–Dym hierarchies and Kaup–Kupershmidt, Camassa–Holm equations naturally arise from motions of plane curves in centro-affine, similarity, affine and fully affine geometries [12,13]. Motions of curves in three-dimensional
centro-affine and affine geometries were also considered [14]. In particular, the Burgers hierarchy describes motion of non-stretching plane curves in similarity geometry [12]. We point out here that no Burgers hierarchy found from motions of curves in other geometries.

In this paper, we are mainly concerned with motions of curves in similarity geometries $P^n$, this problem has been involved in Sapiro and Tannenbaum [15]. The isometry group of similarity geometry is a composition of Euclidean motion and the dilatation. For instance, the corresponding Lie algebras of the isometry groups are generated by $\{\partial_x, \partial_y, x\partial_y, y\partial_x\}$ for $n = 2$ [16] and $\{\partial_x, \partial_y, x\partial_y, x\partial_x, y\partial_y, y\partial_x, y\partial_x, x\partial_y + y\partial_x + u\partial_y\}$ for $n = 3$. All geometric quantities are invariant under the isometry groups. A nice fact is that we can define curvatures and arc-length of the similarity geometry [12], they are characterized by differential invariants and invariant one form of the isometry group. More precisely, let $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ be the curvatures in Euclidean space $\mathbb{R}^n$, then they are differential invariants of $n$-dimensional Euclidean motion. One can readily verify that $\kappa_i = \kappa_{i+2}/k_i^2$, $\kappa_i = k_i/k_1$, $i = 2, \ldots, n-1$, are differential invariants under the similarity motion, we define them to be curvatures in $n$-dimensional similarity geometry, where $s$ is the arc-length of a curve in Euclidean space. Also, $d\theta = k_1 ds$ is an invariant one-form, where $\theta$ is the angle between the tangent and a fixed direction, we define it to be the arc-length of a curve in $P^n$. After that we can define its frame vectors $\tau_i$ in terms of Euclidean’s $t_i$: $\tau_i = \kappa_i t_i$, $i = 1, 2, \ldots, n$. Using them we can represent a geometric motion in the form

$$\gamma_t = \sum_{i=1}^{n} A_i \tau_i,$$

where $A_i$ are the velocities along $\tau_i$ and depend on the curvatures $\kappa_i$ and their derivatives with respect to the arc-length $\theta$. By differentiating (1) with respect to time $t$, we obtain time evolution for vector fields $\tau_i$, $i = 1, 2, \ldots, n$,

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix} = U \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix},$$

By the Frenet–Serret formulas in Euclidean geometry,

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 \\ 0 & -\kappa_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & -\kappa_{n-1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix},$$

one obtains the Frenet–Serret formulas in the similarity geometry $P^n$

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix} = V \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix},$$

with $V$ an $n \times n$ matrix given as follows. The zero curvature condition

$$V_t - U_\theta + [V, U] = 0,$$

gives the equations for the curvatures $\kappa_i$, $i = 1, 2, \ldots, n-1$.

The outline of this paper is as follows: In Section 2, for completeness we study motion of plane curves in $P^2$ by choosing tangent and normal vectors slightly different from [12]. Motions of curves in $P^3$ and $P^n$ ($n > 3$) are discussed respectively in Sections 3 and 4. Section 5 is a concluding remarks on this work.

2. Curves in $P^2$ and the Burgers hierarchy

We denote the tangent, normal, arc-length and curvature in Euclidean space $\mathbb{R}^2$ respectively by $t, n, s$ and $\kappa$. The Frenet–Serret formulas read

$$\begin{pmatrix} t \\ n \end{pmatrix}_s = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} t \\ n \end{pmatrix}.$$
The curvature and arc-length in $P^2$ are given respectively by $\kappa = \kappa_1/\kappa^2$ and $d\theta = \kappa ds$, tangent and normal vectors of a curve in $P^2$ are $\tau_1 = \gamma_0$ and $\tau_2 = n/\kappa$. Via a direct computation we have the Frenet–Serret formulas in $P^2$

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} -\kappa & 1 \\ -1 & -\kappa \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}. \tag{7}$$

Now motion of curves in $P^2$ is specified by

$$\gamma_t = At_1 + Bt_2. \tag{8}$$

We relate it to the Euclidean motion

$$\gamma_t = fn + gt, \tag{9}$$

where $f = B/\kappa$, $g = A/\kappa$. The time variation of the perimeter $L = \int d\theta = \int \kappa ds$ is

$$\frac{dL}{dt} = \int \left( \kappa_t + \frac{s_t}{s} \kappa \right) ds.$$  

Using the formulas in Euclidean geometry [1,2]

$$\kappa_t = fss + sgs + \kappa^2f, \quad s_t = s(g_s - \kappa f),$$

we have

$$\frac{dL}{dt} = \int (f_t + \kappa g_t) ds.$$  

So the inextensible motion means

$$f_t + \kappa g_t = \text{const.} = a,$$

i.e.,

$$A = -B_0 + \kappa B - a. \tag{10}$$

The time evolution for $\tau_1$ and $\tau_2$ is

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} A_0 - \kappa A - B & a \\ -a & A_0 - \kappa A - B \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}. \tag{11}$$

The compatibility condition between (7) and (11) gives the equation for the curvature

$$\kappa_t = -(A_0 - \kappa A - B)_0 = [B_0 - (\kappa B)_0 + \kappa(-B_0 + \kappa B - a) + B]_0 = (\Omega^2 + 1)B_0 - a\partial_0,$$

after using (10), where $\Omega = \partial_0 - \kappa - \kappa_0 \partial_0^{-1}$ is the recursion operator of the Burgers equation. Setting $a = 1$, $B = \kappa_0$, one gets the third-order Burgers equation

$$\kappa_t = \kappa_{000} - 3(\kappa \kappa_0)_0 + 3\kappa_0^2 \kappa_0.$$  

Taking $B_0 = \Omega^{m-2} \kappa_0$, $m \geq 2$, we get the Burgers hierarchy

$$\kappa_t = (\Omega^m + \Omega^{m-2}) \kappa_0 - a\kappa_0,$$

which can be linearized to be

$$\mu_t = \frac{\partial^m \mu}{\partial \Omega^m} + \frac{\partial^{m-2} \mu}{\partial \Omega^{m-2}} - a\mu,$$

through the Cole–Hopf transformation $\kappa = -\mu_0/\mu$.  

3. Curves in $P^3$ and Burgers-mKdV hierarchy

In the case of curves in $P^3$, the Frenet–Serret formulas read

$$
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix}
= \begin{pmatrix}
-k & 1 & 0 \\
-1 & -k & \tau \\
0 & -\tau & -k
\end{pmatrix}
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix},
$$

(13)

where $\kappa$ and $\tau$ are respectively the curvature and torsion in $P^3$, they are related to the Euclidean curvature $\kappa$ and torsion $\tau$ by

$$
\kappa = \frac{\kappa_3}{\kappa^3}, \quad \tau = \frac{\tau}{\kappa}.
$$

The motion is now described by

$$
\gamma_t = A\tau_1 + B\tau_2 + C\tau_3,
$$

(14)

where $\tau_i$, $i = 1, 2, 3$ are related to the Euclidean tangent $t$, normal $n$ and binormal $b$ by

$$
\tau_1 = \frac{t}{\kappa}, \quad \tau_2 = \frac{n}{\kappa}, \quad \tau_3 = \frac{b}{\kappa}.
$$

It is possible to relate (14) with the Euclidean motion in $\mathbb{R}^3$ [2,3]

$$
\gamma_t = Wt + Un + Vb,
$$

(15)

with $W = A/\kappa$, $U = B/\kappa$ and $V = C/\kappa$. Using the formulas in Euclidean space [2]

$$
t_i = (U_i - \tau V + \kappa W)n_i + (V_i + \tau U)b_i,
$$

and

$$
\kappa_i = (n_i, \tau_i) = (n_i, t_i) - \frac{s_i}{s}\kappa,
$$

we obtain

$$
\kappa_i + \frac{s_i}{s}\kappa = (n_i, t_i),
$$

which gives time variation of the perimeter $L = \int d\theta = \int \kappa ds$

$$
\frac{dL}{dt} = \int (\kappa_i + \frac{s_i}{s}\kappa) ds = \int \left[ \frac{1}{\kappa}(U_i - \tau V + \kappa W)_i - \frac{\tau}{\kappa}(V_i + \tau U) \right] d\theta = \int [(U_i - \tau V + \kappa W)_i - \tau(V_i + \tau U)] d\theta
$$

\[= \int [(B_0 - \kappa B - \tau C + A)_i - \tau(C_0 - \kappa C + \tau B)] d\theta.\]

So the inextensibility condition means

$$\int \tau(C_0 - \kappa C + \tau B) d\theta = 0, \quad (B_0 - \kappa B - \tau C + A)_i = \tau(C_0 - \kappa C + \tau B).$$

(16)

The time evolution of the frame is

$$
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix}
= \begin{pmatrix}
F_1 & G_1 & H_1 \\
-G_1 & F_1 & H_2 \\
-H_1 & -H_2 & F_1
\end{pmatrix}
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix},
$$

(17)

where

$$
F_1 = A_0 - \kappa A - B,
G_1 = B_0 - \kappa B - \tau C + A,
H_1 = C_0 - \kappa C + \tau B,
H_2 = (C_0 - \kappa C + \tau B)_0 + \tau(B_0 - \kappa B - \tau C + A).$$
The compatibility condition between (13) and (17) implies \( \tilde{k} \) and \( \tilde{\tau} \) satisfying

\[
\begin{pmatrix}
\tilde{\tau} \\
\tilde{k}
\end{pmatrix} = 
\begin{pmatrix}
\Omega_1 \tilde{\tau} & \Omega_1 (\partial_0 \tilde{k} - \tilde{\tau}) \\
\Omega_2 (\partial_0 \tilde{\tau} - \tilde{k}) + \partial_0 & -\Omega_2 (\partial_0 \tilde{k} - \tilde{\tau}) + \partial_0 - \tilde{k}
\end{pmatrix}
\begin{pmatrix}
B \\
C
\end{pmatrix},
\]

where \( \Omega_1 = \frac{3}{2} \tilde{\tau}^2 + 3 \tilde{\tau} + \tilde{\tau}_0 \tilde{\tau}_0^{-1} \tilde{\tau} + 1 \) and \( \Omega_2 = \partial_0 - \tilde{k} - \tilde{k}_0 \tilde{\tau}_0^{-1} \) are respectively the recursion operators of the mKdV and Burgers equation. In particular, setting

\[
C = -\tilde{\tau}, \quad B = -\tilde{k}, \quad A = \tilde{k}_0 - \tilde{k}^2 - \frac{3}{2} \tilde{\tau}^2,
\]

so that (16) holds. (18) then becomes

\[
\tilde{\tau}_t + \tilde{k}_{000} + \frac{3}{2} \tilde{\tau}_0 \tilde{\tau}_0^{-1} \tilde{\tau}_0 = 0,
\]

\[
\tilde{k}_t + \tilde{k}_{000} - 3(\tilde{k}\tilde{k}_0)_0 + 3\tilde{k}^2\tilde{k}_0 + \frac{3}{2}(\tilde{\tau}_0 \tilde{\tau}_0 - 3(\tilde{\tau}_0)_0 + \tilde{k}_0 = 0.
\]

This system is integrable. In fact, letting \( \tilde{k} = -\mu_0/\mu \), \( \mu \) satisfies

\[
\mu_t + \mu_{000} + \frac{3}{2} (\tilde{\tau}_0 \mu_0 + \mu_0 = 0.
\]

Therefore the motion of curves in \( P^3 \) gives the Burgers-mKdV hierarchy. Noting that for a certain kind of curve satisfying \( \tilde{k} = -\tilde{\tau}_0/\tilde{\tau} \), the system (19) reduces to the mKdV equation (the first one of (19)).

4. Curves in \( P^n \) and a generalization of the Burgers-mKdV hierarchy

Similarly, we have the Frenet–Serret formulas in \( P^n \)

\[
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\vdots \\
\tau_n
\end{pmatrix} = 
\begin{pmatrix}
-z_1 & 1 & 0 \\
-1 & -z_1 & \ddots \\
0 & -z_2 & \ddots \\
& \ddots & \ddots \\
0 & -z_{n-1} & -z_1
\end{pmatrix}
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\vdots \\
\tau_n
\end{pmatrix},
\]

where the curvatures \( z_i \) of curves in \( P^n \), \( i = 2, \ldots, n-1 \), are related to the Euclidean’s curvatures by \( z_1 = k_1/s/k_1^2 \), \( z_i = k_i/k_1, i = 2, 3, \ldots, n-1 \) and the arc-length is given by \( d\theta = k_1 ds \). The curve motion in \( P^n \) is described by (1). The inextensibility condition is

\[
(A_{2,0} - z_1 A_2 + A_1 - z_2 A_3)_{00} = z_2 (A_{3,0} - z_1 A_3 + z_2 A_2 - z_3 A_4),
\]

\[
\int z_2 (A_{3,0} - z_1 A_3 + z_2 A_2 - z_3 A_4) d\theta = 0.
\]

The time evolution for the frame vectors \( \tau_i \) is

\[
\tau_{ij} = (A_{i,0} - z_1 A_i) \tau_i + \sum_{j=1}^{n} M_{ij} \tau_j, \quad i = 1, 2, \ldots, n,
\]

where \( n \times n \) matrix \( M_{ij} \) are given recursively by

\[
M_{i+1,j} = \frac{1}{z_i} (M_{ij} + \tilde{z}_j M_{i-1,j} - z_j M_{i,j+1} + \tilde{z}_j M_{i,j-1}), \quad j > i + 1,
\]

\[
M_{ij} = -M_{ji},
\]

\[
M_{i2} = A_{2,0} - z_1 A_2 + A_1 - z_2 A_3,
\]

\[
M_{i1} = A_{1,0} - z_1 A_1 + z_2 A_2 - z_3 A_4, \quad 3 \leq j \leq n-1,
\]

\[
M_{10} = A_{0,0} - z_1 A_0 + z_2 A_2 - z_3 A_4.
\]
where 
\[ \hat{z}_j = \begin{cases} 
1, & j = 1 \\
0, & j \neq 1.
\end{cases} \]

The compatibility condition between (20) and (22) gives the system for the curvatures \( \zeta_i, i = 1, 2, \ldots, n-1, \)
\[ \begin{align*}
\zeta_{i, t} &= -(A_{i, \theta} - \zeta_i A_1 - A_2)_\theta, \\
\zeta_{i, \theta} &= M_{i, j+1} \zeta_j - \zeta_{i+1} M_{i, j+2} + \hat{z}_i (M_{i, j+1}), \
& \quad i = 2, 3, \ldots, n-1.
\end{align*} \]

(23)

Similar to the previous section, this systems can be written as
\[ \begin{pmatrix} 
\zeta_1 \\
\zeta_2 \\
\vdots \\
\zeta_n
\end{pmatrix}_t = M
\begin{pmatrix} 
A_1 \\
A_2 \\
\vdots \\
A_n
\end{pmatrix}_\theta. \]

For \( n = 4, \) the matrix \( M = (M_{ij})_{3 \times 3} \) is given explicitly by
\[ \begin{align*}
M_{11} &= \left( \partial_{\theta} - \zeta_i - \zeta_i \partial_{\phi}^{-1} \right) \left( \partial_{\phi} - \partial_{\theta} \zeta_i - \zeta_i^2 \right) + \partial_{\phi}, \\
M_{12} &= -\left( \partial_{\theta} - \zeta_i - \zeta_i \partial_{\phi}^{-1} \right) \left( \partial_{\phi} \zeta_i + \zeta_i (\partial_{\theta} \zeta_i) \right) + \partial_{\theta} - \zeta_i, \\
M_{13} &= \left( \partial_{\theta} - \zeta_i - \zeta_i \partial_{\phi}^{-1} \right) \zeta_i \partial_{\phi}, \\
M_{14} &= \partial_{\phi} \zeta_i + \zeta_i \partial_{\phi}^{-1} \zeta_i \partial_{\phi} + \zeta_i \partial_{\phi}^{-1}, \\
M_{22} &= \partial_{\theta} \zeta_i + \zeta_i \partial_{\phi}^{-1} \zeta_i \partial_{\phi} + \zeta_i \partial_{\phi}^{-1}, \\
M_{23} &= \left( \partial_{\theta} - \zeta_i - \zeta_i \partial_{\phi}^{-1} \right) \zeta_i \partial_{\theta}, \\
M_{24} &= \partial_{\phi} \zeta_i + \zeta_i \partial_{\phi}^{-1} \zeta_i \partial_{\phi} + \zeta_i \partial_{\phi}^{-1}, \\
M_{32} &= \partial_{\theta} \zeta_i + \zeta_i \partial_{\phi}^{-1} \zeta_i \partial_{\phi} + \zeta_i \partial_{\phi}^{-1}, \\
M_{33} &= \partial_{\theta} \zeta_i + \zeta_i \partial_{\phi}^{-1} \zeta_i \partial_{\phi} + \zeta_i \partial_{\phi}^{-1}.
\end{align*} \]

Setting
\[ A_1 = \zeta_{i, \theta} - \zeta_i^2 - \frac{3}{2} \zeta_i, \quad A_2 = -\zeta_i, \quad A_3 = -\zeta_i, \quad A_4 = 0, \]
we get the system
\[ \begin{align*}
\zeta_{1, t} &= -\zeta_{1, \theta} + 3 \zeta_{i, \theta} + 3 \zeta_i^2 \zeta_{i, \theta} + \frac{3}{2} (\zeta_i \zeta_i^2) \theta - \zeta_{1, \theta}, \\
\zeta_{2, t} &= -\zeta_{2, \theta} + \frac{3}{2} \zeta_i^2 \zeta_{i, \theta} + \frac{3}{2} \zeta_i^2 \zeta_{i, \theta} + \zeta_i (\zeta_i \zeta_i) \theta + \frac{3}{2} \zeta_i^2 \zeta_{i, \theta} - \zeta_{2, \theta}, \\
\zeta_{3, t} &= \left[ -\zeta_i^2 (\zeta_i \zeta_i + \zeta_i \zeta_i) \theta - \zeta_i^2 \zeta_i \zeta_i \theta - \frac{1}{2} \zeta_i^2 \zeta_i^2 + \frac{1}{2} \zeta_i^2 \zeta_i^2 + \frac{1}{2} \zeta_i^2 \zeta_i^2 \right] \theta - 2 \zeta_i \zeta_i \zeta_i \zeta_i \theta - \zeta_i^2 \zeta_i \zeta_i \theta.
\end{align*} \]

which is a three-component generalization to the integrable system (19).

5. Concluding remarks

In this paper, we have carried out a study on motions of curves in similarity geometries \( P^n \) \((n \geq 2)\). It has been shown that the motions of inextensible curves in \( P^2, P^3 \) and \( P^n (n > 3) \) yield Burgers hierarchy, Burgers\-mKdV hierarchy and a multi-component generalization of these hierarchies respectively. This is compared with the motions of curves in \( R^2, R^3 \) and \( R^n (n > 3) \) respectively yield the mKdV hierarchy, Schrödinger hierarchy and a multi-component generalization of mKdV\-Schrödinger hierarchies.

In [10], Nakayama realized the full AKNS scheme in a hyperboloid in \( M^{1,1} \). By a projection into \( D^1 \), we obtain the hyperbolic geometry in a Klein model. Hence AKNS scheme can be realized as motions of curves in the three-
dimensional hyperbolic geometry. So it is interesting to study motions of curves in the higher-dimensional hyperbolic geometry, surely it will yield a generalization to the AKNS scheme.

Recently, motion of discrete curves has been discussed by several authors [17–19]. They have shown that the Ablowitz–Ladik hierarchy describes motion of discrete plane curves in Euclidean space [17,18]. So it is of interest to study motion of curves in similarity geometries, a discrete Burgers hierarchy will be obtained. The results will be published elsewhere.

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